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MHV rules for Higgs plus multi-gluon amplitudes

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Abstract: We use tree-level perturbation theory to show how non-supersymmetric one-loop scattering amplitudes for a Higgs boson plus an arbitrary number of partons can be constructed, in the limit of a heavy top quark, from a generalization of the scalar graph approach of Cachazo, Svrček and Witten. The Higgs boson couples to gluons through a top quark loop which generates, for large m_t, a dimension-5 operator $H \operatorname{tr} G_{\mu\nu}G^{\mu\nu}$. This effective interaction leads to amplitudes which cannot be described by the standard MHV rules; for example, amplitudes where all of the gluons have positive helicity. We split the effective interaction into the sum of two terms, one holomorphic (selfdual) and one anti-holomorphic (anti-selfdual). The holomorphic interactions give a new set of MHV vertices — identical in form to those of pure gauge theory, except for momentum conservation — that can be combined with pure gauge theory MHV vertices to produce a tower of amplitudes with more than two negative helicities. Similarly, the anti-holomorphic interactions give anti-MHV vertices that can be combined with pure gauge theory anti-MHV vertices to produce a tower of amplitudes with more than two positive helicities. A Higgs boson amplitude is the sum of one MHV-tower amplitude and one anti-MHV-tower amplitude. We present all MHV-tower amplitudes with up to four negative-helicity gluons and any number of positive-helicity gluons (NNMHV). These rules reproduce all of the available analytic formulae for Higgs + n-gluon scattering ($n \leq 5$) at tree level, in some cases yielding considerably shorter expressions.

Keywords: QCD, Higgs Physics, Jets, Hadronic Colliders.

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Contents

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1. Introduction

Since the interpretation of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and QCD as a topological string propagating in twistor space [1] (at least at tree level), there has been a flurry of activity. Very notably, a new set of 'MHV rules' has been proposed for QCD [2], which take the place of ordinary Feynman rules, and lump many Feynman vertices into single color-ordered 'MHV vertices'. These MHV vertices are off-shell continuations of the maximally helicity-violating (MHV) n-gluon scattering amplitudes of Parke and Taylor [3].

Written in terms of spinor inner products [4], they are composed entirely of the 'holomorphic' products $\langle i\, j\rangle$ fashioned from right-handed (undotted) spinors, rather than their anti-holomorphic partners $[i\, j]$,

$$A_n(1^+,\ldots,p^-,\ldots,q^-,\ldots,n^+) = \frac{\langle p\, q\rangle^4}{\langle 1\, 2\rangle\,\langle 2\, 3\rangle\cdots\langle n-1,\, n\rangle\,\langle n\, 1\rangle}\,. \tag{1.1}$$

Here p and q are the only gluons with negative helicity. The MHV vertices (1.1), with a suitable definition for $\langle i\, j\rangle$ when the momenta k_i or k_j are off shell [2], are then connected with scalar-type propagators, which bear the helicity of the intermediate leg but no Lorentz indices. In twistor space, where the anti-holomorphic spinors $\tilde\lambda_{i,\dot\alpha}$ are traded for their Fourier-transform variables $\mu_i^{\dot\alpha} = -i\partial/\partial\tilde\lambda_{i,\dot\alpha}$, each MHV vertex is localized on a line. The lines are connected through the off-shell propagators.

The CSW approach has been extended to amplitudes with fermions [5]. New tree-level gauge-theory results were obtained in this approach for non-MHV amplitudes involving gluons [6, 7], and gluons, fermions and scalars [5, 8, 9].

Although the topological string appears to mix gauge theory with conformal supergravity at the loop level [10], the MHV rules nevertheless work at the one-loop level in supersymmetric Yang-Mills theories (SYM). Brandhuber, Spence and Travaglini [11] used MHV rules to reproduce the series of one-loop n-gluon MHV amplitudes in $\mathcal{N}=4$ SYM, previously computed via unitarity cuts [12]. Very recently, this approach was shown to work also for the same amplitudes in $\mathcal{N}=1$ SYM [13, 14]. On the other hand, the twistor-space structure of both sets of amplitudes seemed to be more complex [15] than the MHV picture would imply. This paradox was resolved by the notion of a 'holomorphic anomaly' [16] due to singularities in the loop-momentum integration. The anomaly for a unitarity cut freezes the phase-space integration, making its evaluation simple [17, 18]. In $\mathcal{N}=4$ SYM, the anomaly can be used to derive algebraic equations for the coefficients of integral functions [18, 19], whose solutions are in agreement with a direct evaluation of the unitarity cuts for 7-gluon amplitudes [20]. In $\mathcal{N}=1$ SYM, differential instead of algebraic equations are obtained [21].

Much of this progress at one loop in massless supersymmetric theories is related to the fact that such theories are 'cut-constructible' [22]. That is, at one loop, intermediate states can be assigned four-dimensional helicities [22], even though the loop-momentum integral must be regulated dimensionally, with $D = 4-2\epsilon$. The 'error' incurred by neglecting the (-2ϵ) components of the momentum in numerators of the cuts can be confined to terms that vanish as $\epsilon\to 0$.

Application of MHV rules to loop amplitudes in *non-supersymmetric* theories seems to be a different matter. This situation is highlighted by the properties of the n-gluon one-loop amplitudes for which all gluons (or all but one) have the same helicity, namely $\mathcal{A}_n^{\text{1-loop}}(1^\pm,2^+,3^+,\ldots,n^+)$ [23, 24, 25, 26]. Such amplitudes vanish in the supersymmetric case, but are nonzero for nonsupersymmetric combinations of massless gluons, fermions or scalars circulating in the loop. They are finite as $\epsilon\to 0$, and in this limit they become cut-free, rational functions of the kinematic invariants. These amplitudes can still be computed from unitarity cuts in D dimensions, by working to $\mathcal{O}(\epsilon)$ or higher, but now the full D-

dimensional set of intermediate states enter [27, 28]. Indeed, in a cut-based construction these amplitudes have support only when the loop momenta are *not* four-dimensional, but point into the (-2ϵ) directions of $(D = 4 - 2\epsilon)$-dimensional space-time. Assigning a four-dimensional helicity to a state circulating around the loop seems unlikely to lead to a correct answer in this case. On the other hand, the quasi-local nature of the $(\pm{+}{+}\cdots{+})$ amplitudes suggests that some of them might be considered fundamental vertices [15], like the tree-level MHV vertices. However, it has not yet been possible to find suitable off-shell continuations, possibly because of the existence of 'anti-holomorphic' spinor products $[i\,j]$ in the numerators of the amplitudes (as we shall review shortly).

A similar problem has plagued attempts to construct MHV rules for gravity [29]. Tree-level gravity amplitudes can be constructed from tree-level gauge-theory amplitudes using low-energy limits of the Kawai-Lewellen-Tye relations in string theory [30]. The gravity amplitudes are the sums of products of pairs of gauge theory amplitudes, but there are additional factors of $s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j$ in the numerator. Because $s_{ij} = \langle i\,j\rangle\,[j\,i]$, anti-holomorphic spinor products also creep in here.

In any case, it is of interest to extend the range of processes that can be treated by MHV-type techniques. One set of processes of much phenomenological interest is the scattering of a single Higgs boson together with a number of quarks and gluons. In fact, production of the Standard Model Higgs boson at hadron colliders such as the Fermilab Tevatron and the CERN Large Hadron Collider is dominated by gluon-gluon fusion, $gg \to H$, through a one-loop diagram containing the top quark in the loop. Precision electroweak data, interpreted within the Standard Model, indicate that the Higgs is considerably lighter than $2m_t \approx 360\,\mathrm{GeV}$; currently $m_H < 260\,\mathrm{GeV}$ at 95% confidence level [31]. In this case, next-to-leading-order QCD computations have shown [32] that it is an excellent approximation to integrate out the heavy top quark, summarizing its effects via the dimension-five operator $H\,\mathrm{tr}\,G_{\mu\nu}G^{\mu\nu}$ [33, 34]. This operator can then be 'dressed' by standard QCD vertices in order to generate Higgs plus multi-parton amplitudes. In this paper, we will provide a set of MHV rules for such amplitudes.

The amplitudes for a Higgs boson plus four gluons ($Hgggg$) were first computed in the heavy top quark approximation by Dawson and Kauffman [35]. Kauffman, Desai and Risal extended these results to the other four-parton processes, $Hggq\bar{q}$ and $Hq\bar{q}Q\bar{Q}$ [36]. More recently, analytic formulae for a Higgs plus up to 5 partons were calculated by Del Duca, Frizzo and Maltoni [37]. Amplitudes for these cases, and those with larger numbers of partons, are also computed numerically by the programs Alpha [38] and MadGraph [39].

An interesting subset of the Higgs plus n-gluon (color-ordered) amplitudes are those for which all gluons have positive helicity,

$$A_n(H, 1^+, 2^+, \dots, n^+) \propto \frac{m_H^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n-1,\, n\rangle\,\langle n\,1\rangle}\,, \tag{1.2}$$

where m_H is the mass of the Higgs boson.[1] There is a strong similarity between the sequence (1.2) and the (leading-color, color-ordered) pure QCD one-loop amplitudes for n

[1] We shall prove this result for all n in appendix B.

positive-helicity gluons [25, 26],

$$A_{n;1}^{\text{1-loop}}(1^+, 2^+, \ldots, n^+) \propto \frac{\sum_{1\leq i_1<i_2<i_3<i_4\leq n} \langle i_1\, i_2\rangle\, [i_2\, i_3]\, \langle i_3\, i_4\rangle\, [i_4\, i_1]}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle n-1,\, n\rangle\, \langle n\, 1\rangle}\,. \tag{1.3}$$

Both sets of amplitudes are generated first at one loop (if we count the top-quark loop in the Higgs case). They are both rational functions of the kinematic variables; i.e. they contain no branch cuts. (For the Higgs case, as we work in the heavy top-quark limit, this statement is rather trivial.) Their collinear and multi-particle factorization properties are very similar, as reflected in the factors in their denominators, $\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle n\, 1\rangle$. The numerator factors are also quite similar, when momentum conservation is taken into account. In the Higgs case, the numerator factor is m_H^4; but this is also expressible in terms of the n gluon momenta as $(\sum_{1\leq i<j\leq n} s_{ij})^2 = (\sum_{1\leq i<j\leq n} \langle i\, j\rangle\, [j\, i])^2$. Thus both the all-plus Higgs and one-loop QCD amplitudes are bi-linear in the anti-holomorphic spinor products $[i\, j]$.

In the one-loop pure-gauge-theory case, a well-motivated but unsuccessful attempt was made [15] to generate the one-loop 5-point amplitude $A_{5;1}^{\text{1-loop}}(-{+}{+}{+}{+})$ from an off-shell continuation of the one-loop 4-point amplitude $A_4^{\text{1-loop}}({+}{+}{+}{+})$, plus the off-shell tree-level MHV vertex $A_3(-{-}{+})$. In the Higgs case, we have made an analogous attempt to generate the Higgs plus three gluon amplitude $A_3(H, -{+}{+})$ from an off-shell continuation of the Higgs plus two gluon amplitude $A_2(H, {+}{+})$, plus the tree-level MHV vertex $A_3(-{-}{+})$. Our attempt failed; it led to results which depended on the choice of the 'reference spinor' in the CSW construction, and thus could not be correct.

For the Higgs case, we shall resolve this problem in a different way, yet still using an MHV-type perturbation theory. As we shall explain in the next section, the crux of our method is to split the $H\,\mathrm{tr}\, G_{\mu\nu}G^{\mu\nu}$ operator into two terms. We will show that MHV rules can be applied to amplitudes generated by one of the two terms. The amplitudes generated by the other term can be obtained by applying 'anti-MHV rules' (or deduced from the first set of amplitudes using parity). The desired $Hgg\ldots g$ amplitude is the sum of one amplitude of each type. The split of $H\,\mathrm{tr}\, G^2$ into two operators can be motivated either by supersymmetry, or by selfduality. The general structure we find can be extended to Higgs amplitudes containing quarks [40], although we shall not do so explicitly in this paper.

We hope that the MHV structure we have uncovered for the Higgs plus multi-parton amplitudes will prove useful for understanding how to apply twistor-MHV methods to one-loop amplitudes in pure QCD, and to tree-level amplitudes in gravity. In the meantime, it allows us to obtain relatively compact expressions for scattering amplitudes of phenomenological interest, such as $gg \to Hggg$. For example, the computation of the cross section for Higgs production via gluon-gluon fusion at nonzero transverse momentum (which may alleviate the QCD background in the $H \to \gamma\gamma$ decay mode [41, 42]) at next-to-leading order (NLO) in α_s requires amplitudes like $gg \to Hgg$; at next-to-next-to-leading order, it requires $gg \to Hggg$. The process $gg \to Hgg$ [43, 44] also appears at leading order (and so $gg \to Hggg$ will be needed at NLO) as a background to production of a Higgs boson via weak boson fusion, $q\bar{Q} \to q'\bar{Q}'W^+W^- \to q'\bar{Q}'H$ [45]. In both cases there are two additional jets, which are used to tag the weak boson fusion production process. The weak boson fusion process is now known at NLO [46, 47], but the gluon-fusion background is currently only available at leading order, with large uncertainties [47]. The background is particularly important if one wants to use azimuthal correlations between the tagging jets to learn about the couplings of the Higgs boson to W pairs [44, 48]. The compact formulae for tree-level Higgs amplitudes should speed up their computation in numerical programs for higher-order cross sections, where they may need to be evaluated very often.

This paper is organized as follows: In section 2, we explain the method in more detail. The new MHV rules are summarized in section 3. Using these rules, we obtain results for all Next-to-MHV and Next-to-Next-to-MHV amplitudes (with up to four negative-helicity gluons and any number of positive-helicity gluons). We show that these results can be used to reproduce all of the available analytic formulae for Higgs + n-gluon scattering ($n \leq 5$) at tree level. We also reproduce the all-n formula (1.2), for the special case where all gluons have the same helicity. It should be straightforward to apply our method to $n > 5$ partons and obtain new analytic results for amplitudes for a Higgs boson plus six partons [40].

In section 4 we consider another example of an effective theory describing gluonic interactions, where the tree-level all-plus helicity amplitude does not vanish, but can again be reconstructed using new MHV rules of the type presented in sections 2 and 3. Our findings are summarized in the Conclusions. There are two appendices. Appendix A summarizes our conventions for color, spinors, helicity and selfduality. Appendix B describes technical details necessary to show the triviality of certain classes of amplitudes. It also contains the recursive construction of the infinite sequence of non-vanishing identical-helicity amplitudes given in eq. (1.2).

2. The model

In the Standard Model the Higgs boson couples to gluons through a fermion loop. The dominant contribution is from the top quark. For large m_t, the top quark can be integrated out, leading to the effective interaction [33, 34],

$$\mathcal{L}_H^{\rm int} = \frac{C}{2}\, H \,{\rm tr}\, G_{\mu\nu} G^{\mu\nu}\,. \tag{2.1}$$

In the Standard Model, and to leading order in α_s, the strength of the interaction is given by $C = \alpha_s/(6\pi v)$, with $v = 246\,{\rm GeV}$.

The MHV or twistor-space structure of the Higgs-plus-gluons amplitudes is best elucidated by dividing the Higgs coupling to gluons, eq. (2.1), into two terms, containing purely selfdual (SD) and purely anti-selfdual (ASD) gluon field strengths,

$$G_{SD}^{\mu\nu} = \frac{1}{2}(G^{\mu\nu} + {}^*G^{\mu\nu})\,, \qquad G_{ASD}^{\mu\nu} = \frac{1}{2}(G^{\mu\nu} - {}^*G^{\mu\nu})\,, \qquad {}^*G^{\mu\nu} \equiv \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}\,. \tag{2.2}$$

This division can be accomplished by considering H to be the real part of a complex field $\phi = \frac{1}{2}(H + iA)$, so that

$$\begin{aligned} \mathcal{L}_{H,A}^{\rm int} &= \frac{C}{2}\Big[H \,{\rm tr}\, G_{\mu\nu} G^{\mu\nu} + iA \,{\rm tr}\, G_{\mu\nu}\, {}^*G^{\mu\nu}\Big] & (2.3)\\ &= C\Big[\phi \,{\rm tr}\, G_{SD\,\mu\nu} G_{SD}^{\mu\nu} + \phi^\dagger \,{\rm tr}\, G_{ASD\,\mu\nu} G_{ASD}^{\mu\nu}\Big]\,. & (2.4)\end{aligned}$$

The key idea is that, due to selfduality, the amplitudes for ϕ plus n gluons, and those for $\phi^\dagger$ plus n gluons, separately have a simpler structure than the amplitudes for H plus n gluons. But because $H = \phi + \phi^\dagger$, the Higgs amplitudes can be recovered as the sum of the ϕ and $\phi^\dagger$ amplitudes.

As another motivation for the split (2.4), note that this interaction can be embedded into an $\mathcal{N} = 1$ supersymmetric effective lagrangian,

$$\mathcal{L}^{\rm int}_{\rm SUSY} = -C \int d^2\theta \; \Phi \, {\rm tr}\, W^\alpha W_\alpha \; - C \int d^2\bar\theta \; \Phi^\dagger \, {\rm tr}\, \overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha} \,. \tag{2.5}$$

Here $G^{\mu\nu}_{SD}$ is the bosonic component of the chiral superfield W_α, and ϕ is the lowest component of the chiral superfield Φ. In appendix A we identify the following helicity assignments:

$$W_\alpha = \{g^-, \lambda^-\}\,, \qquad \Phi = \{\phi, \psi^-\}\,, \tag{2.6}$$

$$\overline{W}^{\dot\alpha} = \{g^+, \lambda^+\}\,, \qquad \Phi^\dagger = \{\phi^\dagger, \psi^+\}\,, \tag{2.7}$$

where $g^\pm$ correspond to gluons with $h = \pm 1$ helicities, $\lambda^\pm$ are gluinos with $h = \pm 1/2$, ϕ and $\phi^\dagger$ are complex scalar fields, and $\psi^\pm$ are their fermionic superpartners. In appendix B we give the full supersymmetric effective lagrangian. (This lagrangian can be generated from a renormalizable, supersymmetric microscopic theory containing a massive top quark/squark chiral multiplet T, coupled to Φ by a Yukawa coupling $\int d^2\theta \Phi T \overline{T}$. Integrating out T produces the interaction (2.5) with a coefficient proportional to the chiral multiplet's contribution to the SYM beta function.)

As in the case of QCD, the fermionic superpartners of the boson ϕ and of the gluons will never enter tree-level processes for ϕ plus n gluons. Thus these bosonic amplitudes must obey supersymmetry Ward identities (SWI) [49], as discussed in appendix B, which help to control their structure. Although we do not explicitly describe Higgs amplitudes involving quarks in this paper, we note that massless quarks can always be added to eq. (2.5) in a supersymmetric fashion, as members of additional chiral multiplets with no superpotential. Then the squarks, as well as the other superpartners, do not enter the amplitudes for a Higgs boson plus multiple quarks and gluons.

We use a standard, trace-based color decomposition for the tree-level amplitudes for n gluons, plus the single colorless field, H, ϕ or $\phi^\dagger$. The full amplitudes can be assembled from color-ordered partial amplitudes A_n, as described in appendix A. Because all amplitudes calculated in this paper are proportional to one power of the constant C appearing in eq. (2.1), we also remove C and powers of the gauge coupling g from the partial amplitudes A_n, via the color decomposition formula (A.1).

As proven in appendix B, a host of helicity amplitudes involving ϕ vanish, namely $\phi g^\pm g^+ g^+ \ldots g^+$. One version of the proof invokes the supersymmetry Ward identities [49], but is valid only for a massless Higgs, $m_H = 0$. The other version uses recursive techniques and the Berends-Giele currents [50], and is valid for any m_H. Thus we know that

$$A_n(\phi, 1^\pm, 2^+, 3^+, \ldots, n^+) = 0\,, \tag{2.8}$$

for all n.

The MHV amplitudes, with precisely two negative helicities, $\phi g^- g^+ \ldots g^+ g^- g^+ \ldots g^+$, are the first non-vanishing ϕ amplitudes. General factorization properties now imply that they have to be extremely simple. They can have no multi-particle poles, because the residue of such a pole would have to be the product of a ϕ amplitude and a pure-glue amplitude, with a total of three external negative-helicity gluons (one is assigned to the intermediate state). At least one of the two product amplitudes must vanish, according to either eq. (2.8) or its pure-glue analogue where the ϕ is deleted. Similarly, only factors of $\langle i\, j\rangle$, not $[i\, j]$, are allowed in the denominator. This property follows from the collinear limit as k_i becomes parallel to k_j; $[i\, j]$ factors are associated with collinear factorization onto vanishing $(n-1)$-point amplitudes of the form (2.8). These same conditions are obeyed by MHV amplitudes in pure QCD; indeed the arguments are identical.

Furthermore, the first few known ϕ amplitudes have precisely the same form as the QCD case — except for the implicit momentum carried out of the process by the Higgs boson. (This momentum makes the Higgs case well-defined on-shell for fewer legs than the pure QCD case.) We have,

$$A_2(\phi, 1^-, 2^-) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,1\rangle} = -\langle 1\,2\rangle^2\,, \tag{2.9}$$

$$A_3(\phi, 1^-, 2^-, 3^+) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle} = \frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,1\rangle}\,, \tag{2.10}$$

$$A_4(\phi, 1^-, 2^-, 3^+, 4^+) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\,. \tag{2.11}$$

This leads to the obvious assertion for *all* 'ϕ-MHV' amplitudes,

$$A_n(\phi, 1^+, 2^+, \ldots, p^-, \ldots, q^-, \ldots, n^+) = \frac{\langle p\,q\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle n-1,\, n\rangle\,\langle n\,1\rangle}\,, \tag{2.12}$$

where only legs p and q have negative helicity. Besides the correct collinear and multi-particle factorization behavior, these amplitudes also correctly reduce to pure QCD amplitudes as the ϕ momentum approaches zero. It should be possible to prove eq. (2.12) recursively, along the lines of the proof in the QCD case [50], perhaps using the additional light-cone recursive currents from ref. [51].

Since the MHV amplitudes (2.12) have an identical form to the corresponding amplitudes of pure Yang-Mills theory, eq. (1.1), we propose that their off-shell continuation is also identical to that proposed in the pure-glue context, in the context of a set of scalar-graph rules [2]. Everywhere the off-shell leg i carrying momentum K_i appears in eq. (2.12), we let the corresponding holomorphic spinor be $\lambda_{i,\alpha} = (K_i)_{\alpha\dot\alpha}\xi^{\dot\alpha}$. Here $\xi^{\dot\alpha}$ is an arbitrary reference spinor, chosen to be the same for all MHV diagrams contributing to the amplitude. (Because anti-holomorphic spinors $\lambda_{i,\dot\alpha}$ do not appear in eq. (2.12), we do not have to discuss their off-shell continuation.)

Figure 1 lays out the MHV structure of the ϕ plus multi-gluon amplitudes. All non-vanishing amplitudes are labelled with circles. The fundamental ϕ-MHV vertices, which coincide with the $\phi g^- g^- g^+ \ldots g^+$ amplitudes (2.12), are the basic building blocks and are labelled by red dots. The result of combining ϕ-MHV vertices with pure-gauge-theory

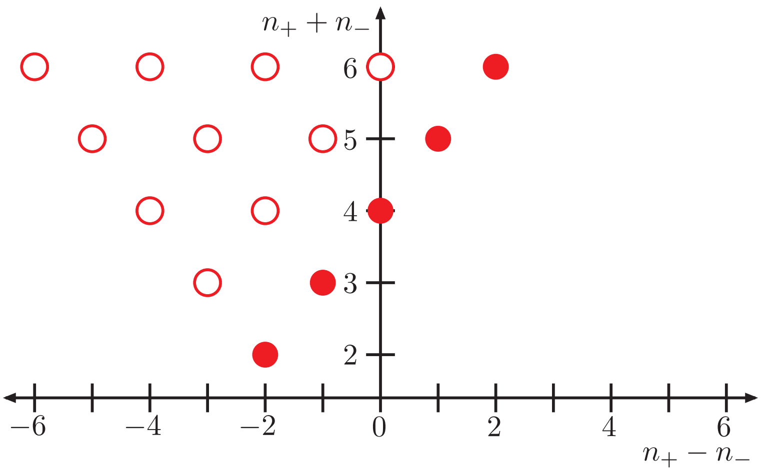

Figure 1: The MHV structure of ϕ plus multi-gluon amplitudes. The number of positive (negative) helicity gluons is n_+ (n_-). The vertical axis labels the total number of gluons, $n_+ + n_-$. The horizontal axis labels the difference $n_+ - n_-$, a measure of the amount of 'helicity violation'. Solid red dots represent fundamental 'ϕ-MHV' vertices. Open red circles represent ϕ amplitudes which are composites, built from the ϕ-MHV vertices plus pure-gauge-theory MHV vertices.

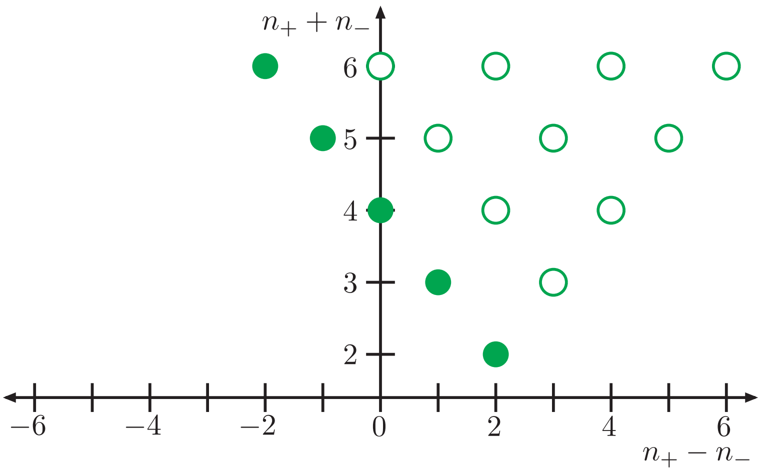

Figure 2: The anti-MHV structure of $\phi^\dagger$ plus multi-gluon amplitudes. The axes are as in figure 1. Solid green dots represent fundamental '$\phi^\dagger$-anti-MHV' vertices, which coincide with the $\phi^\dagger g^+ g^+ g^- \ldots g^-$ amplitudes. Open green circles represent $\phi^\dagger$ amplitudes which are composites, built from the $\phi^\dagger$-anti-MHV vertices plus pure-gauge-theory anti-MHV vertices. These amplitudes can also be obtained from the ϕ amplitudes by parity, which exchanges $\langle i\,j\rangle \leftrightarrow [j\,i]$ and reflects points across the vertical axis, $n_+ - n_- \to -(n_+ - n_-)$.

MHV vertices is to produce amplitudes with more than two negative helicities. These amplitudes are represented as red open circles. Each MHV diagram contains exactly one ϕ-MHV vertex; the rest are pure-gauge-theory MHV vertices. The vertices are combined with scalar propagators. The MHV-drift is always to the left and upwards. Collectively, these amplitudes form the holomorphic (or MHV) tower of accessible amplitudes.

The corresponding amplitudes for $\phi^\dagger$ are shown in figure 2. They can be obtained by applying parity to the ϕ amplitudes. For practical purposes this means that we compute

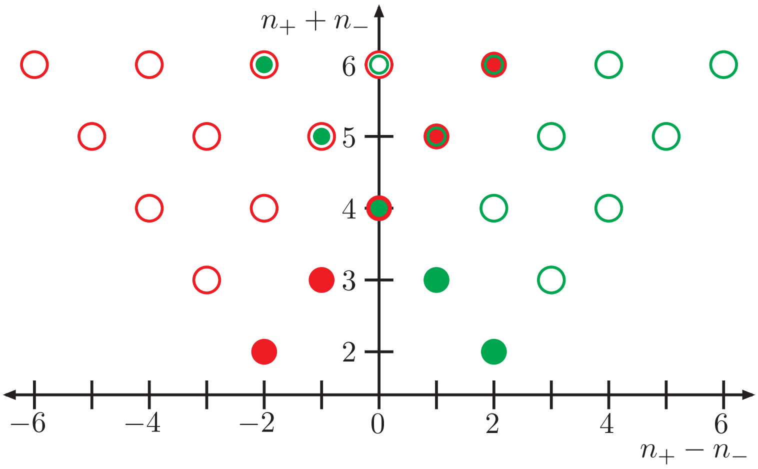

Figure 3: The structure of Higgs plus multi-gluon amplitudes obtained by combining the MHV tower for $\phi + n$ gluons and the anti-MHV tower of $\phi^\dagger + n$ gluon amplitudes. The axes are as in figure 1. Note that the point at $n_+ + n_- = 2$, $n_+ - n_- = 0$, $H \to g^+ g^-$, vanishes by angular momentum conservation. The MHV tower from figure 1 is shown in red. The anti-MHV tower from figure 2 is shown in green. Amplitudes for the scalar Higgs are obtained by adding the ϕ and $\phi^\dagger$ amplitudes.

with ϕ, and reverse the helicities of every gluon. Then we let $\langle i\, j\rangle \leftrightarrow [j\, i]$ to get the desired $\phi^\dagger$ amplitude. The set of building-block amplitudes are therefore anti-MHV. Furthermore, the amplitudes with additional positive-helicity gluons are obtained by combining with anti-MHV gauge theory vertices. The anti-MHV-drift is always to the right and upwards. Collectively, these amplitudes form the anti-holomorphic (or anti-MHV) tower of accessible amplitudes.

To obtain amplitudes for the real Higgs boson with gluons, we merely add the ϕ and $\phi^\dagger$ amplitudes. The allowed helicity states are shown in figure 3 and are composed of both holomorphic and anti-holomorphic structures.

3. MHV rules and applications

The new MHV rules for computing Higgs plus n-gluon scattering amplitudes can be summarized as follows:

1. For the ϕ couplings, everything is exactly like CSW [2] (except for the momentum carried by ϕ).

2. For $\phi^\dagger$, we just apply parity. That is, we compute with ϕ, and reverse the helicities of every gluon. Then we let $\langle i\, j\rangle \leftrightarrow [j\, i]$ to get the desired $\phi^\dagger$ amplitude.

3. For H, we add the ϕ and $\phi^\dagger$ amplitudes.

These rules generate a set of amplitudes with all the correct collinear and multi-particle factorization properties, as follows from the same type of argument as in the pure gauge theory case [2]. In addition, the rules can easily be used to reproduce all of the available analytic formulae for Higgs + n-gluon scattering ($n \leq 5$) at tree level. (In some cases the agreement was checked numerically.) In some instances they generate considerably shorter expressions. To make things even more efficient, one can use a recursive version of the rules, along the lines suggested by ref. [7].

As a by-product, we also obtain the amplitudes for a pseudoscalar Higgs boson A plus multiple gluons, where A couples to gluons via the $A\, G_{\mu\nu}\, {}^*G^{\mu\nu}$ interaction in eq. (2.3). The minimal supersymmetric Standard Model contains such a field. If A is light enough, and the top quark dominates the loop, this effective interaction is a good approximation. Then to construct $A_n(A, 1, 2, \ldots, n)$, in step 3 of the rules we merely need to take the difference instead of the sum of the corresponding ϕ and $\phi^\dagger$ amplitudes.

For the ϕ plus n-gluon amplitudes, we can consider a twistor space $(\lambda_1, \lambda_2, \mu^{\dot 1}, \mu^{\dot 2})$, for each of the n gluons [1], by replacing the anti-holomorphic spinor coordinates $\tilde\lambda_{i,\dot\alpha}$ by their Fourier transforms, $\mu_i^{\dot\alpha} = -i\partial/\partial\tilde\lambda_{i,\dot\alpha}$. In doing this tranformation, we leave the momentum of the massive scalar untouched. Then the argument that the MHV QCD amplitudes are localized on a line [1] extends trivially to the ϕ amplitudes (2.12). The recoiling momentum of the ϕ particle enters the overall momentum-conserving delta function, but this does not affect the localization of the Fourier transform,

$$\begin{aligned}
A(\lambda_i, \mu_i) &= \int \prod_{i=1}^{n} d\tilde\lambda_i \exp(i\mu_i\tilde\lambda_i) A(\lambda_i) \delta\left(k_\phi + \sum_{i=1}^{n} k_i\right) \\
&= \int d^4x A(\lambda_i) \int \prod_{i=1}^{n} d\tilde\lambda_i \exp(i\mu_i\tilde\lambda_i) \exp\left[ix^{\dot\alpha\alpha}\left((k_\phi)_{\alpha\dot\alpha} + \sum_{i=1}^{n} \lambda_{i,\alpha}\tilde\lambda_{i,\dot\alpha}\right)\right], \\
&= \int d^4x \exp(ix \cdot k_\phi) A(\lambda_i) \prod_{i=1}^{n} \delta(\mu_i + x\lambda_i)\,.
\end{aligned} \tag{3.1}$$

The ϕ amplitudes with n_- negative-helicity gluons are similarly localized on networks of $(n_- - 1)$ intersecting lines in twistor space.

3.1 NMHV amplitudes $A_n(\phi, \ldots, m_1^-, \ldots, m_2^-, \ldots, m_3^-, \ldots)$

We start by deriving the Next-to-MHV (NMHV) amplitude $A_n(\phi, m_1^-, m_2^-, m_3^-)$. From now on we will suppress the dots for positive-helicity gluons in the MHV tower of amplitudes. The two topologically distinct diagrams are shown in figure 4. Each of these diagrams is drawn for a fixed arrangement of negative-helicity gluons, such that ϕ is followed by m_1. To obtain the full NMHV amplitude we need to sum over the three cyclic permutations of m_1, m_2 and m_3, denoted by $C(m_1, m_2, m_3)$. The full NMHV amplitude is given by,

$$A_n(\phi, m_1^-, m_2^-, m_3^-) = \frac{1}{\prod_{l=1}^{n} \langle l,\, l+1\rangle} \sum_{i=1}^{2} \sum_{C(m_1,m_2,m_3)} A_n^{(i)}(m_1, m_2, m_3)\,, \tag{3.2}$$

where the common standard denominator is factored out for convenience. We label the gluon momenta as k_i (where i is defined modulo n) and introduce the composite (off-shell) momentum,

$$q_{i+1,j} = k_{i+1} + k_{i+2} + \cdots + k_j\,. \tag{3.3}$$

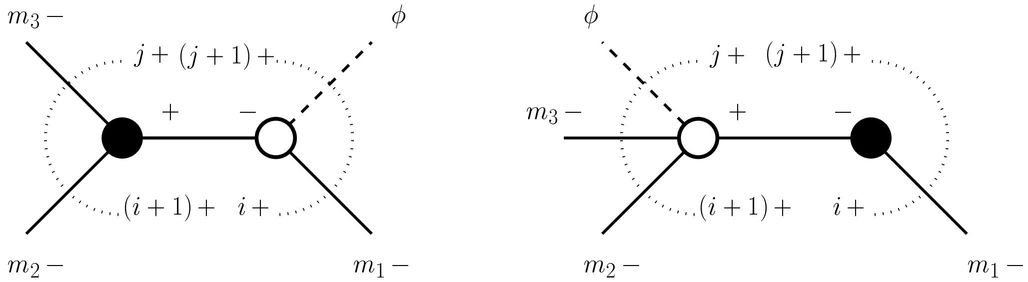

Figure 4: Tree diagrams with MHV vertices which contribute to the NMHV amplitude $A_n(\phi, \ldots, m_1^-, \ldots, m_2^-, \ldots, m_3^-, \ldots)$. The scalar ϕ is represented by a dashed line and negative-helicity gluons, g^-, by solid lines. Positive-helicity gluons g^+ emitted from each vertex are indicated by dotted semicircles, with labels showing the bounding g lines in each MHV vertex.

Note that the momentum of ϕ, k_ϕ, does not enter the sum. In particular, $q_{i+1,i} = -k_\phi$. As usual, the off-shell continuation of the helicity spinor is defined as [2],

$$(\lambda_{i+1,j})_\alpha = (q_{i+1,j})_{\alpha\dot{\alpha}}\, \xi^{\dot{\alpha}}\,, \tag{3.4}$$

where $\xi^{\dot{\alpha}}$ is a reference spinor that can be chosen arbitrarily.

Following the organisational structure of ref. [8], the contributions of the individual diagrams in figure 4 are,

$$\begin{aligned} A_n^{(1)}(m_1, m_2, m_3) &= \sum_{i=m_1}^{m_2-1} \sum_{j=m_3}^{m_1-1} \frac{\langle m_2\, m_3\rangle^4 \, \langle m_1^- | \not{q}_{i+1,j} | \xi^-\rangle^4}{D(i, j, q_{i+1,j})}\,, \\ A_n^{(2)}(m_1, m_2, m_3) &= \sum_{i=m_1}^{m_2-1} \sum_{j=m_3}^{m_1-1} \frac{\langle m_2\, m_3\rangle^4 \, \langle m_1^- | \not{q}_{j+1,i} | \xi^-\rangle^4}{D(i, j, q_{j+1,i})}\,, \end{aligned} \tag{3.5}$$

where

$$D(i, j, q) = \langle i^- | \not{q} | \xi^-\rangle \langle (j+1)^- | \not{q} | \xi^-\rangle \langle (i+1)^- | \not{q} | \xi^-\rangle \langle j^- | \not{q} | \xi^-\rangle \frac{q^2}{\langle i,\, i+1\rangle\, \langle j,\, j+1\rangle}\,. \tag{3.6}$$

In the summation over j, it should be understood that the maximum value is taken modulo n. In other words, when $m_1 = 1$, the upper limit is $0 \equiv n$, but when $m_1 = 2$, the upper limit is 1. Note that diagrams of the second type vanish when there are no positive-helicity gluons emitted from the right hand vertex. In this case, $j + 1 = m_1 = i$ and $q_{j+1,i} = q_{m_1}$. These diagrams are automatically killed by the $\langle m_1^- | q_{j+1,i}$ factor in the numerator.

In distinction with ref. [2], we leave the reference spinor ξ arbitrary and specifically do not set it to be equal to one of the momenta in the problem. This has two advantages. First, we do not introduce unphysical singularities for gluonic amplitudes (for generic points in phase space); and second, it allows a powerful numerical check of gauge invariance (which all of our amplitudes satisfy).

The amplitude 3.2 describes all amplitudes coupling ϕ to 3 negative-helicity gluons and any number of positive-helicity gluons. In particular, it describes the $\phi \to ---$ and $\phi \to +---$ amplitudes. These amplitudes only receive contributions from the MHV tower of amplitudes and are therefore also the amplitudes for $H \to ---$ and $H \to +---$.

3.1.1 $H \to ---$

In this case, we can take $m_1 = 1$, $m_2 = 2$ and $m_3 = 3$. The second class of diagrams in figure 4 collapses since there are not enough gluons to prevent the right hand vertex vanishing. Hence $A_3^{(2)} = 0$. From the first class of diagrams, $A_3^{(1)}$, three individual diagrams survive. (Our counting includes the cyclic permutations of m_1, m_2 and m_3, as required in eq. (3.2).)

3.1.2 $H \to +---$

In this case, we can take $m_1 = 2$, $m_2 = 3$ and $m_3 = 4$. Seven individual diagrams survive, as can be seen from figure 4, including appropriate cyclic permutations. Five of them are of the first type, $A_4^{(1)}$, where ϕ couples directly to one on-shell negative-helicity gluon; two are of the second type, $A_4^{(2)}$, where ϕ couples directly to two negative-helicity gluons.

In both cases, we have checked, with a help of a symbolic manipulator, that our results are ξ-independent (gauge invariant) and agree numerically with the known analytic formulae,

$$A_3(H, 1^-, 2^-, 3^-) = -\frac{m_H^4}{[1\,2]\,[2\,3]\,[3\,1]}\,, \tag{3.7}$$

$$\begin{aligned} A_4(H, 1^+, 2^-, 3^-, 4^-) = &\frac{\langle 3^-|\not{k}_H|1^-\rangle^2\,\langle 2\,4\rangle^2}{s_{124}s_{12}s_{14}} + \frac{\langle 4^-|\not{k}_H|1^-\rangle^2\,\langle 2\,3\rangle^2}{s_{123}s_{12}s_{23}} + \frac{\langle 2^-|\not{k}_H|1^-\rangle^2\,\langle 3\,4\rangle^2}{s_{134}s_{14}s_{34}} - \\ &-\frac{\langle 2\,4\rangle}{\langle 12\rangle[23][34]\langle 41\rangle}\left(s_{23}\frac{\langle 2^-|\not{k}_H|1^-\rangle}{[4\,1]} + s_{34}\frac{\langle 4^-|\not{k}_H|1^-\rangle}{[1\,2]} - s_{234}\langle 24\rangle\right). \end{aligned} \tag{3.8}$$

where $k_H = k_\phi$.

3.2 NNMHV amplitudes $A_n(\phi, \ldots, m_1^-, \ldots, m_2^-, \ldots, m_3^-, \ldots, m_4^-, \ldots)$

The Next-to-Next-to-MHV (NNMHV) amplitudes follow from diagrams with three MHV vertices. The skeleton diagram in figure 5 shows our labelling conventions for gluons.

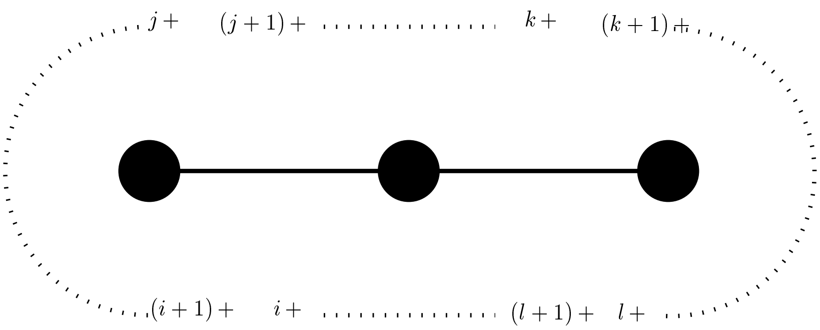

Figure 5: Skeleton diagram showing the labelling of n gluons for amplitudes A_n with four negative-helicity gluons. Positive-helicity gluons g^+ emitted from each vertex are indicated by dotted lines with labels showing the bounding g^+ lines in each MHV vertex.

There are thirteen topologically distinct diagrams in this case, shown in figure 6. As before, each of these diagrams is drawn for the fixed arrangement of negative-helicity gluons, such that ϕ is followed by m_1. To obtain the full NNMHV amplitude we need to sum over all cyclic permutations, $C(m_1, m_2, m_3, m_4)$. The resulting total amplitude is given by,

$$A_n(\phi, m_1^-, m_2^-, m_3^-, m_4^-) = \frac{1}{\prod_{l=1}^n \langle l,\, l+1\rangle} \sum_{i=1}^{13} \sum_{C(m_1,m_2,m_3,m_4)} A_n^{(i)}(m_1, m_2, m_3, m_4)\,. \quad (3.9)$$

The contributions of the first five diagrams in figure 6 are,

$$\begin{aligned}
A_n^{(1)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1} \sum_{j=m_4}^{k} \sum_{i=m_2}^{m_3-1} \sum_{l=m_1}^{m_2-1} \frac{\langle m_3\, m_4\rangle^4 \, \langle m_2^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{l+1,k}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{l+1,k})}\,, \\
A_n^{(2)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1} \sum_{j=m_3}^{m_4-1} \sum_{i=m_1}^{m_2-1} \sum_{l=m_1}^{i} \frac{\langle m_2\, m_3\rangle^4 \, \langle m_4^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{l+1,k}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{l+1,k})}\,, \\
A_n^{(3)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1} \sum_{j=m_4}^{k} \sum_{i=m_3}^{m_4-1} \sum_{l=m_1}^{m_2-1} \frac{\langle m_2\, m_3\rangle^4 \, \langle m_4^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{l+1,k}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{l+1,k})}\,, \\
A_n^{(4)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1} \sum_{j=m_2}^{m_3-1} \sum_{i=m_1}^{m_2-1} \sum_{l=m_1}^{i} \frac{\langle m_3\, m_4\rangle^4 \, \langle m_2^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{l+1,k}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{l+1,k})}\,, \\
A_n^{(5)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1} \sum_{j=m_3}^{m_4-1} \sum_{i=m_2}^{m_3-1} \sum_{l=m_1}^{m_2-1} \frac{\langle m_2\, m_4\rangle^4 \, \langle m_3^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{l+1,k}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{l+1,k})}\,.
\end{aligned} \quad (3.10)$$

A comment is in order concerning the boundary values in the sums over external gluons in eq. (3.10) for cases where it is possible to have no external gluon legs emitted from a given vertex (and in a given range). Consider, for example, the first diagram in figure 6. There are no negative-helicity gluons, g^-, emitted upwards from the middle vertex, and the number of g^+ gluons, n_+, can be zero or non-zero. The summations over j and k in $A_n^{(1)}$ should read:

$$\sum_{k=m_4}^{m_1-1} \sum_{j=m_4}^{k} = \sum_{k=m_4+1}^{m_1-1} \sum_{j=m_4}^{k-1} + \sum_{k=m_4}^{m_1-1} \bigg|_{j=k}\,, \quad (3.11)$$

where the first term (the double sum) corresponds to $n_+ > 0$, and the second term takes into account the case of $n_+ = 0$, or no g^+ gluons emitted upwards from the middle vertex.

The effective propagator DD is defined by,

$$DD(i,j,q_1,k,l,q_2) = \chi(j,k,q_1,q_2)\chi(l,i,q_2,q_1)\, D(i,j,q_1)\, D(k,l,q_2)\,, \quad (3.12)$$

where D is given in eq. (3.6), and where

$$\begin{aligned}
\chi(j,k,q_1,q_2) &= 1 && \text{if } j \neq k, \\
&= \frac{\langle j,\, j+1\rangle \,\langle \xi^+|\not{q}_1\not{q}_2|\xi^-\rangle}{\langle (j+1)^-|\not{q}_1|\xi^-\rangle\langle j^-|\not{q}_2|\xi^-\rangle} && \text{if } j = k\,.
\end{aligned} \quad (3.13)$$

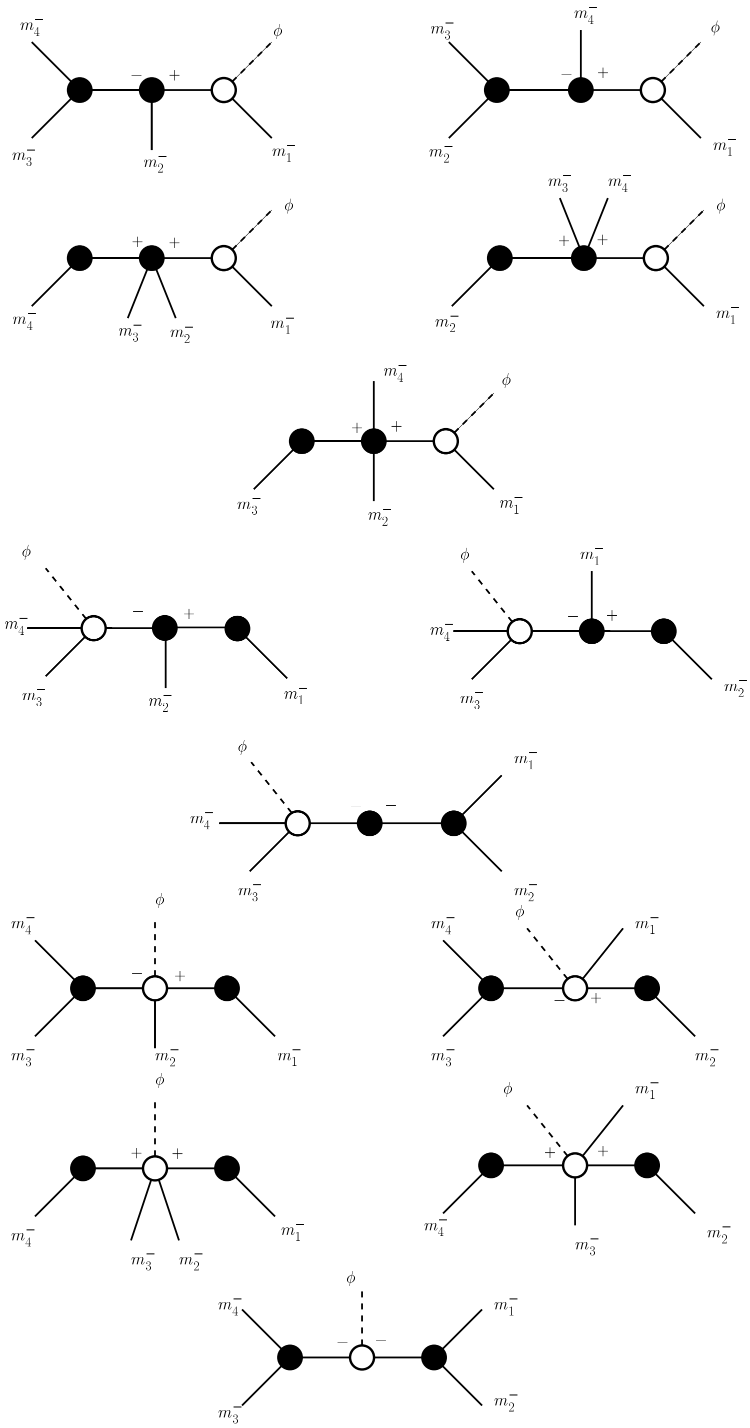

Figure 6: NNMHV tree diagrams contributing to the amplitude $A_n(\phi, g^-_{m_1}, g^-_{m_2}, g^-_{m_3}, g^-_{m_4})$.

The contributions of diagrams 6, 7 and 8 in figure 6 read,

$$A_n^{(6)}(m_1, m_2, m_3, m_4) = \sum_{k=m_4}^{m_1-1} \sum_{j=m_4}^{k} \sum_{i=m_2}^{m_3-1} \sum_{l=m_1}^{m_2-1} \frac{\langle m_3\, m_4\rangle^4 \, \langle m_2^-|\not{q}_{j+1,i}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(k, l, q_{k+1,l}, i, j, q_{j+1,i})},$$

$$A_n^{(7)}(m_1,m_2,m_3,m_4) = \sum_{k=m_1}^{m_2-1}\sum_{j=m_4}^{m_1-1}\sum_{i=m_2}^{m_3-1}\sum_{l=m_2}^{i} \frac{\langle m_3\, m_4\rangle^4 \, \langle m_1^-|\not{q}_{j+1,i}|\xi^-\rangle^4 \langle m_2^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(k,l,q_{k+1,l},i,j,q_{j+1,i})},$$

$$A_n^{(8)}(m_1,m_2,m_3,m_4) = \sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_2}^{m_3-1}\sum_{l=m_2}^{i} \frac{\langle m_3\, m_4\rangle^4 \, \langle \xi^+|\not{q}_{j+1,i}\,\not{q}_{k+1,l}|\xi^-\rangle^4 \, \langle m_1\, m_2\rangle^4}{DD(k,l,q_{k+1,l},i,j,q_{j+1,i})}. \tag{3.14}$$

Finally, for the last five diagrams in figure 6 we have,

$$A_n^{(9)}(m_1,m_2,m_3,m_4) = \sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_2}^{m_3-1}\sum_{l=m_1}^{m_2-1} \frac{\langle m_3\, m_4\rangle^4 \, \langle m_2^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{k+1,l})},$$

$$A_n^{(10)}(m_1,m_2,m_3,m_4) = \sum_{k=m_1}^{m_2-1}\sum_{j=m_4}^{m_1-1}\sum_{i=m_2}^{m_3-1}\sum_{l=m_2}^{i} \frac{\langle m_3\, m_4\rangle^4 \, \langle m_1^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_2^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{k+1,l})},$$

$$A_n^{(11)}(m_1,m_2,m_3,m_4) = \sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_3}^{m_4-1}\sum_{l=m_1}^{m_2-1} \frac{\langle m_2\, m_3\rangle^4 \, \langle m_4^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{k+1,l})},$$

$$A_n^{(12)}(m_1,m_2,m_3,m_4) = \frac{1}{2}\sum_{k=m_1}^{m_2-1}\sum_{j=m_4}^{m_1-1}\sum_{i=m_3}^{m_4-1}\sum_{l=m_2}^{m_3-1} \frac{\langle m_1\, m_3\rangle^4 \, \langle m_4^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_2^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{k+1,l})},$$

$$A_n^{(13)}(m_1,m_2,m_3,m_4) = \frac{1}{2}\sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_2}^{m_3-1}\sum_{l=m_2}^{i} \frac{\langle m_3\, m_4\rangle^4 \, \langle \xi^+|\not{q}_{i+1,j}\,\not{q}_{k+1,l}|\xi^-\rangle^4 \, \langle m_1\, m_2\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{k+1,l})}. \tag{3.15}$$

Note that the expressions for the last two diagrams, $A_n^{(12)}$ and $A_n^{(13)}$, contain a factor of 1/2. This factor is necessary to take into account the fact that, out of the four cyclic permutations $C(m_1,m_2,m_3,m_4)$ in eq. (3.9), only two give inequivalent diagrams; the remaining two double up the result.

3.2.1 $H \to ----$

Our general NNMHV expressions (3.9) can be applied to the simple case with no positive-helicity gluons, $n_+ = 0$. Only diagrams 1, 2 and 13 survive — all others give zero contribution because there are not enough gluons to prevent one of the vertices from vanishing. We checked numerically that our result is gauge invariant and agrees with the known expression,

$$A_4(H,1^-,2^-,3^-,4^-) = A_4(\phi,1^-,2^-,3^-,4^-) = \frac{m_H^4}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,1]}\,. \tag{3.16}$$

3.2.2 $H \to +----$

The amplitude for $n_+ = 1$,

$$A_5(H,1^+,2^-,3^-,4^-,5^-) = A_5(\phi,1^+,2^-,3^-,4^-,5^-)\,, \tag{3.17}$$

can be obtained by setting $m_1 = 2$, $m_2 = 3$, $m_3 = 4$ and $m_4 = 5$ in eq. (3.9). The result is again gauge invariant, and we have checked that it agrees numerically with eq. (B.2) in ref. [37].

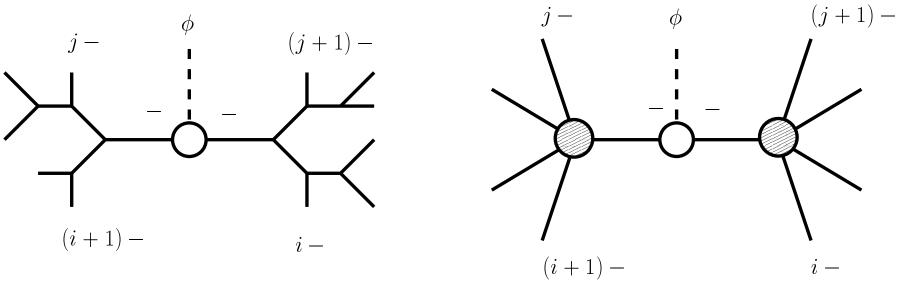

Figure 7: Two representations for $A_n(\phi, 1^-, 2^-, 3^-, \ldots, n^-)$ in the MHV-rules approach. In (a), we illustrate how attaching only negative-helicity gluons requires just three-point MHV vertices. In (b), the shaded circle represents the coupling of an off-shell gluon to many on-shell negative helicity gluons, which is obtained by summing MHV graphs of the type shown in (a).

3.3 $H \to - - - \cdots -$

In appendix B.3 we present a recursive construction of all n-gluon amplitudes with gluons of the same helicity: $A_n(\phi^\dagger, 1^+, 2^+, \ldots, n^+)$, with parity conjugates $A_n(\phi, 1^-, 2^-, \ldots, n^-)$. This derivation makes use of the Berends-Giele off-shell currents [50].

In this section we will derive the same amplitudes using the MHV rules. The two derivations will turn out to be almost identical; nevertheless, we believe that it is instructive to demonstrate the all-orders-in-n agreement between the results derived in the standard approach (appendix B.3) and in the MHV-rules approach (this section).

In the MHV-rules approach, the 'minus-only' amplitudes,

$$A_n(H, 1^-, 2^-, 3^-, \ldots, n^-) = A_n(\phi, 1^-, 2^-, 3^-, \ldots, n^-)\,, \tag{3.18}$$

are constructed by attaching $n-2$ of the tree-level three-point MHV vertices, $A_3(+--)$, to the ϕ-MHV two-gluon vertex, $A_2(\phi, --)$, in all possible ways, as depicted in figure 7a. Each of the two showers of three-point MHV vertices shown in figure 7 can be represented by an off-shell effective vertex,

$$V_n(g_1^{+*}, g_2^-, g_3^-, \ldots, g_n^-) = \frac{p_1^2}{[\xi\, 2]\,[n\, \xi]}\, \frac{(-1)^n}{[2\,3]\,[3\,4] \cdots [n-1,\, n]}\,, \tag{3.19}$$

where it is understood that only the g^+ leg is off shell and hence $p_1^2 \neq 0$. The expression (3.19) was derived in ref. [52] using the MHV rules of CSW [2], and proved by induction. It is also similar in spirit to the all-plus Berends-Giele [50] off-shell current (B.18) used in appendix B. The factor $(-1)^n$ on the right hand side of eq. (3.19) reflects our conventions for the $[i\, j]$ spinor product, eq. (A.5).

Attaching the off-shell V-vertices on both sides of the amplitude $A_2(\phi, -, -)$, we get the following expression:

$$A_n(\phi, 1^-, 2^-, 3^-, \ldots, n^-) = \frac{(-1)^n}{[1\,2]\,[2\,3] \cdots [n\,1]}\, \mathcal{A}\,, \tag{3.20}$$

where $\mathcal{A}$ is the sum

$$\mathcal{A} = -\sum_{1 \le i < j \le n} \frac{[i,\, i+1]\,[j,\, j+1]}{[i\, \xi]\,[\xi,\, i+1]\,[j\, \xi]\,[\xi,\, j+1]}\, \langle \lambda_{i+1,j}\, \lambda_{j+1,i} \rangle^2\,. \tag{3.21}$$

The sum (3.21) (or more precisely, its hermitian conjugate) is computed in appendix B.3, in eqs. (B.25)–(B.28). The result is $\mathcal{A} = m_H^4$. We conclude that

$$A_n(\phi, 1^-, 2^-, 3^-, \dots, n^-) = \frac{(-1)^n\, m_H^4}{[1\,2]\,[2\,3]\cdots[n\,1]}\,, \tag{3.22}$$

and, similarly, the parity conjugate amplitude is

$$A_n(\phi^\dagger, 1^+, 2^+, 3^+, \dots, n^+) = \frac{m_H^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,, \tag{3.23}$$

in agreement with the results derived in appendix B.3 using standard methods.

3.4 Amplitudes in the overlap of two towers

So far, we have described amplitudes that receive contributions from the MHV tower of amplitudes. Transcribing these results to amplitudes that lie in the anti-MHV tower is straightforward. We reverse the helicities of every gluon and let $\langle i\,j\rangle \leftrightarrow [j\,i]$ throughout. For more complicated objects like $\langle m_1^- |\not{q}|\xi^-\rangle$, we make the replacements

$$\begin{aligned}\langle m_1^- |\not{q}\,|\xi^-\rangle &\to \langle m_1^+ |\not{q}\,|\xi^+\rangle\,,\\ \langle m_1^+ |\not{q}_1\not{q}_2\,|\xi^-\rangle &\to -\langle m_1^- |\not{q}_1\not{q}_2\,|\xi^+\rangle\,.\end{aligned} \tag{3.24}$$

Now we would like to describe amplitudes that lie in the overlap of the two towers.

3.4.1 $H \to ++--$

This is the simplest amplitude which receives contributions from both the MHV and the anti-MHV towers. The first contribution is simply the ϕ-MHV amplitude (2.11), and the second one is its parity conjugate $\phi^\dagger$-anti-MHV. In total we have,

$$A_4(H, 1^+, 2^+, 3^-, 4^-) = \frac{\langle 3\,4\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle} + \frac{[1\,2]^4}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,1]}\,, \tag{3.25}$$

which is the correct result.

3.4.2 $H \to ++---$

This Higgs-plus-five-gluons amplitude receives contributions from both towers. The contribution from the ϕ-MHV tower is an NMHV amplitude $A_5(\phi, 1^+, 2^+, 3^-, 4^-, 5^-)$ of the type constructed in eq. (3.2), where we set $m_1 = 3$, $m_2 = 4$ and $m_3 = 5$. In total there are 7 contributions of type $A_5^{(1)}$ and 4 of type $A_5^{(2)}$. The contribution from the $\phi^\dagger$-anti-MHV tower is the simple anti-MHV diagram $A_5(\phi^\dagger, 1^+, 2^+, 3^-, 4^-, 5^-)$, which is the parity conjugate of eq. (2.12),

$$A_5(\phi^\dagger, 1^+, 2^+, 3^-, 4^-, 5^-) = -\frac{[1\,2]^4}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]}\,. \tag{3.26}$$

The final result,

$$A_5(H, 1^+, 2^+, 3^-, 4^-, 5^-) = A_5(\phi, 1^+, 2^+, 3^-, 4^-, 5^-) + A_5(\phi^\dagger, 1^+, 2^+, 3^-, 4^-, 5^-)\,, \tag{3.27}$$

is gauge invariant and agrees numerically with eq. (B.3) of ref. [37].

3.5 The soft Higgs limit

For the case of a massless Higgs boson, we can consider the kinematic limit where the Higgs momentum goes to zero. In this limit, because of the form of the $HG_{\mu\nu}G^{\mu\nu}$ interaction, the Higgs boson behaves like a constant, namely the gauge coupling. Hence the Higgs-plus-n-gluon amplitudes should become proportional to the pure-gauge-theory amplitudes,

$$\mathcal{A}_n(H, \{k_i, \lambda_i, a_i\}) \longrightarrow (\text{const.}) \times \frac{\partial}{\partial g}\mathcal{A}_n(\{k_i, \lambda_i, a_i\}), \qquad \text{as } k_H \to 0, \tag{3.28}$$

for any helicity configuration. Taking into account the gauge coupling factors in the color decomposition (A.1), eq. (3.28) becomes, for the partial amplitudes,

$$A_n(H, 1, 2, 3, \ldots, n) \longrightarrow (n-2)A_n(1, 2, 3, \ldots, n) \qquad \text{as } k_H \to 0. \tag{3.29}$$

It is interesting to see how the soft Higgs limit is partitioned between the ϕ and $\phi^\dagger$ amplitudes. Clearly the ϕ-MHV amplitudes (2.12) become precisely equal to the corresponding pure-gauge MHV amplitudes (1.1). From figure 4 it is then apparent that the NMHV amplitudes will approach twice the corresponding pure-gauge MHV amplitudes, because the field ϕ can be attached to either of the two MHV vertices in the gauge theory case. From figure 6 there is a factor of 3 in the NNMHV limit. More generally,

$$A_n(\phi, 1, 2, 3, \ldots, n) \longrightarrow (n_- - 1)A_n(1, 2, 3, \ldots, n) \qquad \text{as } k_\phi \to 0. \tag{3.30}$$

Parity tells us that the $\phi^\dagger$ amplitudes obey,

$$A_n(\phi^\dagger, 1, 2, 3, \ldots, n) \longrightarrow (n_+ - 1)A_n(1, 2, 3, \ldots, n) \qquad \text{as } k_\phi \to 0. \tag{3.31}$$

Summing eqs. (3.30) and (3.31) and using $n_+ + n_- = n$, we recover the soft Higgs limit (3.29).

For $n_- \neq n_+$, the ϕ and $\phi^\dagger$ limits are different. This result is a bit curious, because as $k_\phi \to 0$, the ϕ and $\phi^\dagger$ interactions become equivalent — the $AG_{\mu\nu}\,{}^*G^{\mu\nu}$ coupling they differ by becomes a total derivative as the field A becomes a constant. But the interactions are apparently not becoming equivalent fast enough to prevent the ϕ and $\phi^\dagger$ amplitudes from having different limits.

In the usual MHV-rules approach to amplitudes in pure gauge theory [2], one considers all tree amplitudes to come from the MHV tower, or sometimes all to come from the anti-MHV-tower. The $k_H \to 0$ limit of the Higgs amplitude construction suggests that one can also consider a mixture of the two towers, where the MHV/anti-MHV content of a given amplitude depends on the number of positive and negative helicity gluons it contains, according to eqs. (3.30) and (3.31).

4. Another effective model

Our general approach to constructing MHV rules can be tested in a second interesting model with the effective operator,

$$O_\Lambda = \frac{1}{\Lambda^2}\,\mathrm{tr}\, G_\mu{}^\nu G_\nu{}^\rho G_\rho{}^\mu\,. \tag{4.1}$$

JHEP12(2004)015

This operator is the unique gauge-invariant, CP-even, dimension-6 operator built solely from gluon fields (after applying equations of motion). Hence it provides a sensible way to characterize possible deviations of gluon self-interactions from those predicted by QCD, such as might be produced by gluon compositeness [53]. The operator O_Λ also may be produced by integrating out a heavy colored fermion (or scalar), albeit with a phenomenologically tiny coefficient. Various ways to probe for such an operator experimentally have been proposed [53, 54, 55].

Here we will consider the deviations that are linear in $1/\Lambda^2$; that is, the set of (color-ordered) amplitudes $A_n^{(\Lambda)}$ arising from one insertion of O_Λ, combined with any number of tree-level QCD interactions. Curiously, at the four-parton level, the amplitudes generated in this way are orthogonal to those of QCD. For example, the four-gluon helicity amplitudes which are produced are those which vanish in tree-level QCD, $(++++)$, $(-+++)$, and the parity conjugates $(+---)$ and $(----)$. At the linearized level, O_Λ does not produce $(--++)$. Thus the interference with tree-level QCD vanishes at this order. Similarly, the $q\bar{q}gg$ amplitudes with only one power of O_Λ produce only $(\mp\pm++)$ and the parity conjugate $(\pm\mp--)$, which vanish in QCD. The five-gluon amplitudes which do interfere with QCD, $(--+++)$ and its parity conjugates, were computed in ref. [55].

The first on-shell n-gluon amplitudes generated by eq. (4.1) are those for four gluons, which read, dropping an overall factor of $(-12i\pi/\Lambda^2)$ [55]:

$$A_4^{(\Lambda)}(1^-,2^-,3^-,4^-) = -\frac{2s_{12}s_{23}s_{13}}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,1]}\,, \tag{4.2}$$

$$A_4^{(\Lambda)}(1^+,2^-,3^-,4^-) = \frac{\langle 2\,3\rangle^2\langle 3\,4\rangle^2\langle 4\,2\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\,, \tag{4.3}$$

$$A_4^{(\Lambda)}(1^-,2^-,3^+,4^+) = 0\,, \tag{4.4}$$

$$A_4^{(\Lambda)}(1^-,2^+,3^-,4^+) = 0\,, \tag{4.5}$$

plus the parity conjugate amplitudes, $A_4^{(\Lambda)}(1^+,2^+,3^+,4^+)$ and $A_4^{(\Lambda)}(1^-,2^+,3^+,4^+)$.

To apply our MHV-rules method to this model, we first rewrite the effective operator (4.1) as the sum of a holomorphic (selfdual) and an anti-holomorphic (anti-selfdual) term,

$$O_\Lambda = \frac{1}{\Lambda^2}\left(\mathrm{tr}\, G_{SD\,\mu}{}^{\nu}\, G_{SD\,\nu}{}^{\rho}\, G_{SD\,\rho}{}^{\mu} + \mathrm{tr}\, G_{ASD\,\mu}{}^{\nu}\, G_{ASD\,\nu}{}^{\rho}\, G_{ASD\,\rho}{}^{\mu}\right)\,. \tag{4.6}$$

The holomorphic interaction, $(G_{SD})^3$, generates amplitudes with a minimum of 3 negative-helicity gluons, such as eqs. (4.2) and (4.3), whereas the anti-holomorphic interaction, $(G_{ASD})^3$, generates amplitudes with a minimum of 3 positive-helicity gluons, such as $(-+++)$ and $(++++)$. (We can prove that the $(G_{SD})^3$ amplitudes with 0 or 1 negative-helicity gluons vanish, along the same lines as the recursive vanishing proof in the Higgs case in appendix B.2, using the structure of the vertex and eqs. (B.20) and (B.21). We assume that the $(G_{SD})^3$ amplitudes with 2 negative-helicity gluons also vanish.) That is, $(G_{SD})^3$ leads to amplitudes with $n_- \geq 3$ and $n_+ \geq 0$, whereas $(G_{ASD})^3$ induces amplitudes with $n_+ \geq 3$ and $n_- \geq 0$, as plotted in figure 8.

JHEP12(2004)015

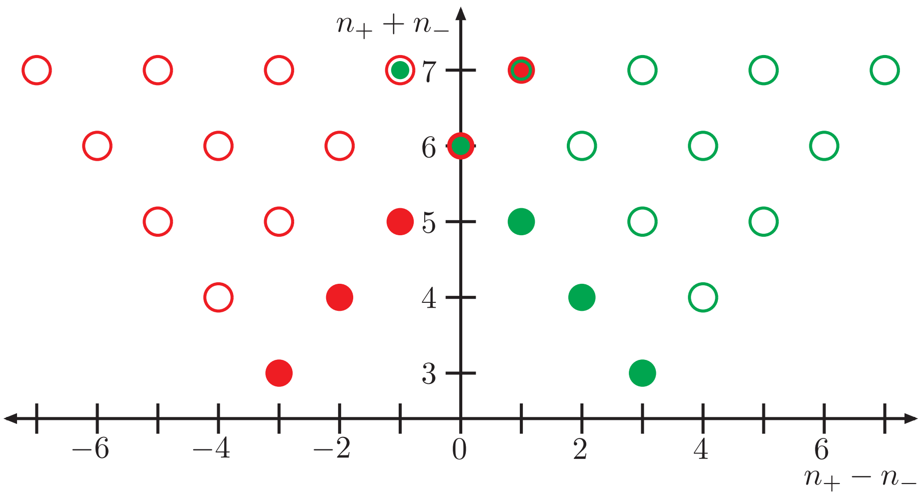

Figure 8: The proposed structure of n-gluon amplitudes induced by the operator $\mathrm{tr}(G^3)$. The MHV tower for $\mathrm{tr}(G^3_{SD})$-induced amplitudes contains pure MHV vertices (closed red circles) plus those amplitudes obtained by adding pure-gauge-theory MHV vertices (open red circles). The anti-MHV tower of $\mathrm{tr}(G^3_{ASD})$-induced amplitudes is obtained from the MHV tower by parity, and is shown in green. The $\mathrm{tr}(G^3)$-induced amplitudes are given by the sum of the two contributions, where they overlap. The two entries with $n_+ + n_- = 3$ represent MHV vertices, but not on-shell scattering amplitudes.

The building blocks induced by the holomorphic $(G_{SD})^3$ interaction are the G^3-MHV vertices with $n_- = 3$ and arbitrary $n_+ \geq 0$:

$$A_n^{(\Lambda)}(1^+, \ldots, i^-, \ldots, j^-, \ldots, k^-, \ldots, n^+) = \frac{(\langle i\, j\rangle \, \langle j\, k\rangle \, \langle k\, i\rangle)^2}{\langle 1\, 2\rangle \, \langle 2\, 3\rangle \cdots \langle n\, 1\rangle}\,, \tag{4.7}$$

where only gluons i, j, k have negative helicity. On shell, these are known to be correct for $n = 4$ and 5 [55], and they have the right factorization properties for $n > 5$ (assuming the vanishing of $(G_{SD})^3$ amplitudes with 2 negative-helicity gluons). We continue the spinor inner products off shell in the by now familiar way [2]. The anti-holomorphic $(G_{ASD})^3$ interaction gives the G^3-anti-MHV vertices with $n_+ = 3$ and arbitrary $n_- \geq 0$, which are the parity conjugates of eq. (4.7).

The construction of the amplitudes induced by $\mathrm{tr}\, G^3$ closely parallels the Higgs case. To build the holomorphic MHV tower we use the G^3-MHV vertices (4.7), combined with standard tree-level MHV vertices. The anti-holomorphic tower is its parity conjugate, built from G^3-anti-MHV vertices combined with pure gauge theory anti-MHV vertices. Figure 8 depicts the two towers. In the overlap region (see figure 8) we add the two terms. In the G^3 case the overlap does not start until the 6-gluon amplitudes, $(---+++)$, plus permutations.

Just as in the Higgs case, this construction has the correct general behavior under collinear and multi-particle factorization. Note that the G^3-MHV vertex (4.7), like the standard MHV vertex (1.1), has no $-- \to -$ or $+- \to +$ collinear singularity for $n > 3$. But it also is completely nonsingular in the collinear limit for $n = 3$, so all the collinear singularities are described by the QCD splitting amplitudes, as required [55]. Other than this consistency test, we have not carried out as extensive checks of the $\operatorname{tr} G^3$ model as we did for the Higgs case, mainly because only a limited number of amplitudes has been computed previously. However, we have tested that the off-shell continuation of the 3-point version of eq. (4.7), namely $\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,1\rangle$, combined with the usual pure-gauge MHV $(--+)$ vertex, successfully reproduces the $(----)$ amplitude given in eq. (4.2).

We can think of the effective interaction (4.1) as being generated by integrating out a massive fermion or scalar loop in a non-supersymmetric theory. However, it is still possible to embed the holomorphic plus anti-holomorphic decomposition (4.6) into an $\mathcal{N} = 1$ supersymmetric interaction,

$$\mathcal{L}^{\rm int} = \frac{1}{\Lambda^2} \int d^2\theta \, \mathrm{tr}[(D^\beta \, W^\alpha) \, W_\beta \, W_\alpha] + \mathrm{h.c.} \tag{4.8}$$

The holomorphic part becomes a superpotential, exactly as in the Higgs model. Hence we again have a supersymmetric completion of an effective interaction, which is an F-term.[2] However, in this case we have not been able to use supersymmetric Ward identities to show vanishings of tree-level amplitudes with less than 2 negative (or less than 2 positive) helicities. In fact, we know that such amplitudes, e.g. eqs. (4.2) and (4.3), do not vanish. The reason why supersymmetric Ward identities cannot be used in this model is that the supersymmetry of the theory is spontaneously broken in the presence of the interactions (4.8). There is a term D^3/Λ^2 in eq. (4.8) which, combined with the D^2 term from the SYM lagrangian, leads the auxiliary D field to develop a non-vanishing vacuum expectation value, $\langle D\rangle \propto \Lambda^2$. Thus the supercharges do not annihilate the vacuum.

5. Conclusions and outlook

In this paper we have constructed and tested a novel set of MHV rules for calculating scattering amplitudes of the massive Higgs boson plus an arbitrary number of gluons. The model which we use to calculate these amplitudes is the tree-level pure gauge theory plus an effective interaction $HG_{\mu\nu}G^{\mu\nu}$. This effective interaction is generated in the heavy top quark limit from the non-supersymmetric Standard Model by integrating out the heavy top-quark loop. To be able to apply MHV rules, we split the interaction into selfdual and anti-selfdual pieces. The MHV rules lead to compact formulae for the Higgs plus multi-parton amplitudes induced at leading order in QCD in the large m_t limit. We presented explicit formulae for the ϕ-plus-n-gluon amplitudes containing up to four negative-helicity gluons, and an arbitrary number of positive-helicity gluons.

This structure may also be useful for going to the next order in QCD — one-loop amplitudes for the Higgs boson plus many partons (or two loops if we count the top-quark loop). At present, such amplitudes are known for up to three partons, namely the processes $Hggg$ and $Hgq\bar{q}$ [56], but the four-parton cases are required for the NLO weak-boson-fusion background computation mentioned in the introduction. One can split the computation

[2] Because the interaction is an F-term, it should not be generated perturbatively in a supersymmetric theory. This is why we took the microscopic theory, with a heavy fermion or scalar in the loop, to be nonsupersymmetric.

into ϕ and $\phi^\dagger$ terms. Some of the helicity amplitudes should then become quite simple — namely, those for which the corresponding tree amplitudes vanish, $A_n(\phi, 1^\pm, 2^+, \ldots, n^+)$. These amplitudes must be rational functions, free of all cuts, by the same argument as in the pure-gauge-theory case [25, 26]. It may well be possible to determine them for all n in a similar fashion, by using recursive or collinear-based arguments.

Apart from being interesting on its own right, as discussed in the introduction, we believe that this model gives important insights into the structure of the MHV rules in generic nonsupersymmetric theories at the loop level. In fact, the two examples of effective theories we have considered suggest a useful generalization.

Consider certain classes of loop diagrams in non-supersymmetric gauge theories, such that loops can be integrated out and represented by higher-dimensional operators in the effective action. The key idea for constructing MHV rules for this effective action, is to split the higher-dimensional operators into holomorphic and anti-holomorphic terms, such that the holomorphic terms can be embedded into a superpotential of a supersymmetric theory. In other words, the holomorphic interactions will involve only chiral superfields W_α and Φ (if matter is present). In components, this means separating selfdual and anti-selfdual components of the field strength. Then we build the MHV tower by combining new MHV vertices from the holomorphic superpotential with the standard tree-level MHV vertices; the anti-MHV tower is built by combining anti-MHV vertices from the anti-holomorphic interactions with the standard anti-MHV vertices.

The MHV-rules construction described in this paper was designed to address effective interactions at tree level. From the perspective of the microscopic theory, our approach enables us to address only massive loops in a non-supersymmetric theory. One of the main points we want to make is that these MHV rules amount to more than adding a new class of vertices to the tree-level rules of ref. [2], in particular, the two towers of MHV and anti-MHV diagrams are crucial for the construction to work. In the soft-Higgs limit, $k_H \to 0$, each tower becomes proportional to the pure-gauge-theory tower, but the constant of proportionality depends on the helicity content of the amplitude. We expect that our findings will be useful in constructing MHV rules also for massless loops in nonsupersymmetric theories.

Acknowledgments

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A. Conventions

A.1 Color

The tree-level Higgs-plus-gluons amplitudes can be decomposed into color-ordered partial amplitudes [35, 37] as

$$\mathcal{A}_n(H, \{k_i, \lambda_i, a_i\}) = iCg^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}})\, A_n(H, \sigma(1^{\lambda_1}, \ldots, n^{\lambda_n}))\,. \quad \text{(A.1)}$$

Here S_n/Z_n is the group of non-cyclic permutations on n symbols, and j^{λ_j} labels the momentum k_j and helicity λ_j of the $j^{\rm th}$ gluon, which carries the adjoint representation index a_i. The T^{a_i} are fundamental representation SU(N_c) color matrices, normalized so that $\mathrm{Tr}(T^aT^b) = \delta^{ab}$. The strong coupling constant is $\alpha_s = g^2/(4\pi)$.

Color-ordering means that, in a computation based on Feynman diagrams, the partial amplitude $A_n(H, 1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n})$ would receive contributions only from planar tree diagrams with a specific cyclic ordering of the external gluons: $1, 2, \ldots, n$. Because the Higgs boson is uncolored, there is no color restriction on how it is emitted. The partial amplitude A_n is invariant under cyclic permutations of its gluonic arguments. It also obeys a reflection identity,

$$A_n(H, n, n-1, \ldots, 2, 1) = (-1)^n A_n(H, 1, 2, \ldots, n-1, n)\,, \tag{A.2}$$

and a dual Ward identity,

$$A_n(H, 1, 2, 3, \ldots, n) + A_n(H, 2, 1, 3, \ldots, n) + \cdots + A_n(H, 2, 3, \ldots, n-1, 1, n) = 0\,. \tag{A.3}$$

These properties mimic those of the corresponding pure-gluon amplitudes where the Higgs is omitted.

A.2 Spinors, helicity and selfduality

We work in Minkowski space with the metric $\eta^{\mu\nu}$ and use the sigma matrices from Wess and Bagger [57], $\sigma^\mu_{\alpha\dot\alpha} = (-1, \tau^1, \tau^2, \tau^3)$, and $(\bar\sigma^\mu)^{\dot\alpha\alpha} = (-1, -\tau^1, -\tau^2, -\tau^3)$, where $\tau^{1,2,3}$ are the Pauli matrices.

In the spinor helicity formalism [4] an on-shell momentum of a massless particle, $k_\mu k^\mu = 0$, is represented as

$$k_{\alpha\dot\alpha} \equiv k_\mu \sigma^\mu_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}\,, \tag{A.4}$$

where λ_α and $\tilde\lambda_{\dot\alpha}$ are two commuting spinors of positive and negative chirality. Spinor inner products are defined by[3]

$$\langle\lambda, \lambda'\rangle = \epsilon_{\alpha\beta}\lambda^\alpha\lambda'^\beta\,, \qquad [\tilde\lambda, \tilde\lambda'] = -\epsilon_{\dot\alpha\dot\beta}\tilde\lambda^{\dot\alpha}\tilde\lambda'^{\dot\beta}\,, \tag{A.5}$$

and a scalar product of two null vectors, $k_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$ and $p_{\alpha\dot\alpha} = \lambda'_\alpha\tilde\lambda'_{\dot\alpha}$, becomes

$$k_\mu p^\mu = -\frac{1}{2}\langle\lambda, \lambda'\rangle[\tilde\lambda, \tilde\lambda']\,. \tag{A.6}$$

We use the shorthand $\langle i\, j\rangle$ and $[i\, j]$ for the inner products of the spinors corresponding to momenta k_i and k_j,

$$\langle i\, j\rangle = \langle\lambda_i, \lambda_j\rangle\,, \qquad [i\, j] = [\tilde\lambda_i, \tilde\lambda_j]. \tag{A.7}$$

For gluon polarization vectors we use

$$\varepsilon^\pm_\mu(k, \xi) = \pm\frac{\langle\xi^\mp|\gamma_\mu|k^\mp\rangle}{\sqrt{2}\langle\xi^\mp|k^\pm\rangle}\,, \tag{A.8}$$

[3] Our conventions for spinor helicities follow refs. [1, 2], except that $[ij] = -[ij]_{CSW}$ as in ref. [58].

where k is the gluon momentum and ξ is the reference momentum, an arbitrary null vector which can be represented as the product of two reference spinors, $\xi_{\alpha\dot\alpha} = \xi_\alpha \tilde\xi_{\dot\alpha}$. We choose the reference momenta for all gluons to be the same, unless otherwise specified. In terms of helicity spinors, $\varepsilon_{\alpha\dot\alpha} = \varepsilon^\mu(\gamma_\mu)_{\alpha\dot\alpha}$, eq. (A.8) takes the form [1],

$$\varepsilon^+_{\alpha\dot\alpha} = \sqrt{2}\frac{\xi_\alpha \tilde\lambda_{\dot\alpha}}{\langle \xi\ \lambda\rangle}\,, \tag{A.9}$$

$$\varepsilon^-_{\alpha\dot\alpha} = \sqrt{2}\frac{\lambda_\alpha \tilde\xi_{\dot\alpha}}{[\tilde\lambda\ \tilde\xi]}\,. \tag{A.10}$$

To simplify the notation, we will drop the tilde-sign over the dotted reference spinor, so that $\xi_{\alpha\dot\alpha} = \xi_\alpha \xi_{\dot\alpha}$.

It follows from eq. (A.8), or eqs. (A.9) and (A.10), that

$$\varepsilon^+_i \cdot \varepsilon^+_j = 0, \qquad \varepsilon^-_i \cdot \varepsilon^-_j = 0\,. \tag{A.11}$$

We define the dual field strength in Minkowski space via

$${}^*G^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}\,, \tag{A.12}$$

and $\epsilon^{0123} = 1 = -\epsilon_{0123}$. The selfdual (SD) part of the field strength is selected via $(\sigma^{\mu\nu})_\alpha^{\ \beta} G_{\mu\nu}$, where

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)\,, \qquad \sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}\,. \tag{A.13}$$

Similarly, the combination $(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} G_{\mu\nu}$ gives rise to the anti-selfdual (ASD) part of the field strength. Here

$$\bar\sigma^{\mu\nu} = \frac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)\,, \qquad \bar\sigma^{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma}\,. \tag{A.14}$$

In general, $G^{\mu\nu}$ can be written as a selfdual plus an anti-selfdual contribution,

$$G^{\mu\nu} = (\sigma^{\mu\nu})_\alpha^{\ \beta}\, g^\alpha_{\ \beta} + (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\, \tilde g^{\dot\beta}_{\ \dot\alpha}\,. \tag{A.15}$$

It is convenient to re-express the field strength in terms of spinor indices as $G_{\alpha\dot\alpha\beta\dot\beta} = \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} G_{\mu\nu}$. Then the decomposition above reads as follows,[4]

$$G_{\alpha\dot\alpha\beta\dot\beta} = \epsilon_{\dot\alpha\dot\beta}\,(g_{\alpha\beta} + g_{\beta\alpha}) - \epsilon_{\alpha\beta}\,(\tilde g_{\dot\alpha\dot\beta} + \tilde g_{\dot\beta\dot\alpha})\,. \tag{A.16}$$

From this one concludes that $\epsilon_{\dot\alpha\dot\beta}$ multiplies the SD-component of the field strength, and $\epsilon_{\alpha\beta}$ multiplies the ASD-component. It then follows, as in ref. [1], that the ASD field strength corresponds to positive-helicity gluons g^+ and the SD component gives negative-helicity gluons g^-. To verify this, note that the linearized field strength

$$G_{\alpha\dot\alpha\beta\dot\beta} = -i(k_{\alpha\dot\alpha}\varepsilon_{\beta\dot\beta} - k_{\beta\dot\beta}\varepsilon_{\alpha\dot\alpha})\,, \tag{A.17}$$

[4] In deriving eq. (A.16) we have used the σ-matrix identities, $\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}(\sigma^{\mu\nu})_\gamma^{\ \delta} = \epsilon_{\dot\beta\dot\alpha}(\epsilon_{\gamma\alpha}\,\delta^\delta_\beta + \epsilon_{\gamma\beta}\,\delta^\delta_\alpha)$, and $\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}(\bar\sigma^{\mu\nu})^{\dot\gamma}_{\ \dot\delta} = \epsilon_{\beta\alpha}(\epsilon_{\dot\beta\dot\delta}\,\delta^{\dot\gamma}_{\dot\alpha} + \epsilon_{\dot\alpha\dot\delta}\,\delta^{\dot\gamma}_{\dot\beta})$.

evaluated for the positive polarization vector from eq. (A.9), becomes proportional to $\tilde{\lambda}_{\dot\alpha}\tilde{\lambda}_{\dot\beta}(\lambda_\alpha\xi_\beta - \lambda_\beta\xi_\alpha)$. Thus it contains only the $\epsilon_{\alpha\beta}$ (ASD) term. Similarly, inserting the negative polarization vector (A.10) into eq. (A.17) leads to a result containing only the $\epsilon_{\dot\alpha\dot\beta}$ (SD) term.

In a supersymmetric theory, the SD field strength $\sigma^{\mu\nu}G_{\mu\nu}$ enters the chiral superfield W_α, and the ASD combination, $\bar\sigma^{\mu\nu}G_{\mu\nu}$, enters the anti-chiral superfield $\overline{W}^{\dot\alpha}$ [57]. Since the SD field strength corresponds to a negative-helicity gluon, we will associate all component fields of *chiral* superfields with *negative* helicity particles, and those of *anti-chiral* superfields with *positive* ones. Hence we have

$$W_\alpha = \{g^-, \lambda^-\}\,, \qquad \Phi = \{\phi, \psi^-\}\,, \tag{A.18}$$
$$\overline{W}^{\dot\alpha} = \{g^+, \lambda^+\}\,, \qquad \Phi^\dagger = \{\phi^\dagger, \psi^+\}\,, \tag{A.19}$$

where $g^\pm$ correspond to gluons with helicities $h = \pm 1$; $\lambda^\pm$ are gluinos with $h = \pm 1/2$; ϕ and $\phi^\dagger$ are complex scalar fields; and $\psi^\pm$ are their fermionic superpartners. If the mass m_H of the chiral superfield vanishes, then $\psi^\pm$ are $h = \pm 1/2$ helicity eigenstates. For nonzero m_H they are chirality, but not helicity, eigenstates.

B. Vanishing of $A_n(\phi, 1^\pm, 2^+, 3^+, \ldots, n^+)$

In this appendix, we demonstrate that for the coupling of ϕ, there is nothing 'more MHV' than the MHV amplitudes. That is, we show that $A_n(\phi, g_1^\pm, g_2^+, g_3^+, \ldots, g_n^+) = 0$. We can do this in two ways, using supersymmetric Ward identities and also more directly, via the Berends-Giele recursion relations and off-shell currents [50]. We also compute the non-vanishing amplitudes $A_n(\phi^\dagger, 1^+, 2^+, 3^+, \ldots, n^+)$ recursively.

B.1 Supersymmetry argument

Since the HGG interaction has a supersymmetric completion (2.5), we can use supersymmetric Ward identities [49] to demonstrate vanishings of certain tree amplitudes. Before proceeding, we write down the full supersymmetric lagrangian,

$$\begin{aligned}\mathcal{L} &= \int d^4\theta\ \Phi^\dagger\Phi + \int d^2\theta\Big[\frac{m_H}{2}\Phi^2 + \frac{1}{4}(1-4C\Phi)\,\mathrm{tr}\,W^\alpha W_\alpha\Big] + \\ &\quad + \int d^2\bar\theta\ \Big[\frac{m_H}{2}\Phi^{\dagger 2} + \frac{1}{4}(1-4C\Phi^\dagger)\,\mathrm{tr}\,\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\Big] \end{aligned}\tag{B.1}$$

$$\begin{aligned} &= F^\dagger F - \partial_\mu\phi^\dagger\partial^\mu\phi - i\bar\psi\bar\partial\!\!\!/\psi + \frac{1}{2}\,\mathrm{tr}\,D^2 - \frac{1}{4}\,\mathrm{tr}\,G_{\mu\nu}G^{\mu\nu} - i\bar\lambda D\!\!\!\!/\lambda + \\ &\quad + \Big\{ m_H\Big(F\phi - \frac{1}{2}\psi^2\Big) - C\Big[-F\,\mathrm{tr}\,\lambda\lambda - 2\sqrt{2}i\psi^\alpha\,\mathrm{tr}\Big(\lambda_\alpha D - (\sigma_{\mu\nu})_\alpha{}^\beta\lambda_\beta\,G^{\mu\nu}_{SD}\Big) + \\ &\qquad\qquad + \phi\,\mathrm{tr}\Big(-2i\bar\lambda D\!\!\!\!/\lambda - G_{SD\,\mu\nu}G^{\mu\nu}_{SD}\Big)\Big] + \mathrm{h.c.}\Big\}. \end{aligned}\tag{B.2}$$

Because the term linear in the auxiliary field D is also linear in the coefficient C, the D-term 'potential' from integrating out D is quadratic in C and may be neglected. On the other hand, the F-term interaction has a linear term,

$$\mathcal{L}_F = -|m_H\phi - C\lambda\lambda|^2 = -m_H^2\phi^\dagger\phi + Cm_H(\phi^\dagger\lambda\lambda + \phi\bar{\lambda}\bar{\lambda}) + \mathcal{O}(C^2). \tag{B.3}$$

To derive the supersymmetry Ward identities we will use the supersymmetry transformations of the on-shell fields in the helicity basis, following ref. [58],

$$[Q(\xi)\,,\,\lambda^+(k)] = -\theta\langle\xi\;k\rangle\,g^+(k)\,, \qquad [Q(\xi)\,,\,\lambda^-(k)] = +\theta[\xi\;k]\,g^-(k)\,, \tag{B.4}$$

$$[Q(\xi)\,,\,g^-(k)] = +\theta\langle\xi\;k\rangle\,\lambda^-(k)\,, \qquad [Q(\xi)\,,\,g^+(k)] = -\theta[\xi\;k]\,\lambda^+(k)\,, \tag{B.5}$$

$$[Q(\xi)\,,\,\phi^\dagger(k)] = -\theta\langle\xi\;k\rangle\,\psi^+(k)\,, \qquad [Q(\xi)\,,\,\phi(k)] = +\theta[\xi\;k]\,\psi^-(k)\,, \tag{B.6}$$

$$[Q(\xi)\,,\,\psi^-(k)] = +\theta\langle\xi\;k\rangle\,\phi(k)\,, \qquad [Q(\xi)\,,\,\psi^+(k)] = -\theta[\xi\;k]\,\phi^\dagger(k)\,. \tag{B.7}$$

Here the operator $Q(\xi)$ is a Lorentz singlet entering a commutative (rather than anticommutative) algebra with all the fields. It is obtained from the standard spinor supercharge by contracting it with a commuting reference spinor ξ and multiplying it by a Grassmann number θ. This defines a commuting singlet operator $Q(\xi)$. In what follows, the anticommuting parameter θ will cancel from the relevant expressions for the amplitudes. The reference spinors, ξ_α and $\xi_{\dot\alpha}$, are arbitrary spinors; they will be fixed below.

The supersymmetry relations involving ϕ and ψ, eqs. (B.6) and (B.7), have been written for the massless Higgs case, $m_H = 0$. In principle, one can extend these relations in such a way that they can also be applied in the case of a massive Higgs boson. However, it turns out that the resulting SWI are not as useful for $m_H \neq 0$, so in the end we will only consider the massless Higgs case.

In order to prove that $A_n(\phi, g_1^\pm, g_2^+, g_3^+, \ldots, g_n^+) = 0$ we consider the following equation:

$$\langle\,0|\,[Q(\xi)\,,\,\phi_k\,g_{k_1}^\pm\,\lambda_{k_2}^+\,g_{k_3}^+\;\ldots\;g_{k_n}^+]\,|\,0\,\rangle = 0\,. \tag{B.8}$$

The right-hand side is zero because, in a theory with unbroken supersymmetry, the supercharge Q annihilates the vacuum.

First, we consider eq. (B.8) for the negative helicity choice $g_{k_1}^-$. We find

$$\begin{aligned} 0 = {}& [\xi\;k]\langle 0|\psi_k^-\,g_{k_1}^-\,\lambda_{k_2}^+\,g_{k_3}^+\;\ldots|0\rangle + \langle\xi\;k_1\rangle\langle 0|\phi_k\,\lambda_{k_1}^-\,\lambda_{k_2}^+\,g_{k_3}^+\;\ldots|0\rangle - \\ & -\langle\xi\;k_2\rangle\langle 0|\phi_k\,g_{k_1}^- g_{k_2}^+\,g_{k_3}^+\;\ldots|0\rangle + [\xi\;k_3]\langle 0|\phi_k\,g_{k_1}^-\lambda_{k_2}^+\,\lambda_{k_3}^+\;\ldots|0\rangle + \cdots\,. \end{aligned} \tag{B.9}$$

The third term on the right-hand side of eq. (B.9) is the amplitude we want to investigate. The remaining terms in eq. (B.9) contain one fermion-antifermion pair. We will now set the reference spinor ξ to be equal to k_1 in order to discard the second term in eq. (B.9). For $m_H = 0$, we can drop the λ-chirality-violating F-terms (B.3). Then chirality conservation implies that λ^+ can only be in the same amplitude with a λ^- (our conventions are that all particles are incoming), which kills the last set of terms. Furthermore, there are no Feynman diagrams which can connect the ψ^- fermion to the λ^+ fermion, because the $\psi\sigma_{\mu\nu}\,\mathrm{tr}\,\lambda\,G_{SD}^{\mu\nu}$ interaction (plus hermitian conjugate) in eq. (B.2) is of the type,

$$\mathcal{L}^{\mathrm{int}} \ni \psi^-\lambda^-g^- + \psi^+\lambda^+g^+\,. \tag{B.10}$$

Thus the first term also vanishes. Because all the fermion-containing terms in eq. (B.9) vanish, we conclude that the amplitude $A_n(\phi, g_1^-, g_2^+, g_3^+, \ldots, g_n^+) = 0$.

We can similarly demonstrate that $A_n(\phi, g_1^+, g_2^+, g_3^+, \ldots, g_n^+) = 0$ by starting with eq. (B.8) and the positive-helicity gluon $g_{k_1}^+$.

At the same time, it is instructive to show that when ϕ is exchanged with $\phi^\dagger$ the amplitudes with less than two negative-helicity gluons can be non-vanishing. Here we will concentrate on amplitudes with one negative-helicity gluon (amplitudes with no negative-helicity gluons are proportional to m_H^4 and vanish in the massless limit we consider at present). We proceed by considering a Ward identity,

$$\langle 0 \,|\, [Q\,,\, \phi_k^\dagger\, g_{k_1}^-\, \lambda_{k_2}^+\, g_{k_3}^+\, \ldots\, g_{k_n}^+] \,|\, 0 \rangle = 0\,. \tag{B.11}$$

This gives us,

$$\begin{aligned} 0 = &-\langle \xi\ k\rangle \langle 0|\psi_k^+\, g_{k_1}^-\, \lambda_{k_2}^+\, g_{k_3}^+\, \ldots |0\rangle + \langle \xi\ k_1\rangle \langle 0|\phi_k^\dagger\, \lambda_{k_1}^-\, \lambda_{k_2}^+\, g_{k_3}^+\, \ldots |0\rangle - \\ &-\langle \xi\ k_2\rangle \langle 0|\phi_k^\dagger\, g_{k_1}^- g_{k_2}^+\, g_{k_3}^+\, \ldots |0\rangle + [\xi\ k_3] \langle 0|\phi_k^\dagger\, g_{k_1}^- \lambda_{k_2}^+\, \lambda_{k_3}^+\, \ldots |0\rangle + \cdots\,. \end{aligned} \tag{B.12}$$

The third term on the right-hand side of eq. (B.12) is the amplitude we want to investigate. The main difference with eq. (B.9) is that now the first term on the right hand side of eq. (B.12) is non-vanishing, since the $\psi^+\lambda^+ g^+$ interaction is allowed due to eq. (B.10). This implies that $A_n(\phi^\dagger, g_1^+, g_2^+, g_3^+, \ldots, g_n^+) \neq 0$.

In the next part of this appendix we recover these results without appealing to supersymmetric Ward identities or setting $m_H = 0$.

B.2 Direct demonstration

We can also show that $A_n(\phi, 1^\pm, 2^+, 3^+, \ldots, n^+)$ vanishes using the Berends-Giele recursion relations and off-shell currents [50]. (For a review, see ref. [58].)

The two-point vertex coupling ϕ to two (off-shell) gluons with outgoing momenta k_1 and k_2 and Lorentz indices μ_1 and μ_2 is

$$V^H_{\mu_1\mu_2}(k_1, k_2) = \eta_{\mu_1\mu_2} k_1 \cdot k_2 - k_{1\,\mu_2} k_{2\,\mu_1} + i\varepsilon_{\mu_1\mu_2\nu_1\nu_2} k_1^{\nu_1} k_2^{\nu_2}\,. \tag{B.13}$$

For $\phi^\dagger$ the sign of the Levi-Civita term would be reversed.

First let us compute the simplest amplitudes $A_2(\phi, 1^\pm, 2^\pm)$ using this vertex. (The opposite-helicity cases vanish using angular-momentum conservation, $A_2(\phi, 1^\pm, 2^\mp) = 0$.) For gluon polarization vectors we use eq. (A.8). From identities (A.11) it follows that only the second and third terms in the vertex (B.13) contribute to $A_2(\phi, 1^\pm, 2^\pm)$. Consider the ratio of their contributions in the positive-helicity case,

$$R^{++} \equiv \frac{i\varepsilon_{\mu_1\mu_2\nu_1\nu_2} \varepsilon^{+,\mu_1} \varepsilon^{+,\mu_2} k_1^{\nu_1} k_2^{\nu_2}}{-\varepsilon_1^+ \cdot k_2\ \varepsilon_2^+ \cdot k_1} = -\frac{1}{4} \frac{\mathrm{tr}[\gamma_5\ \not\varepsilon_1^+\ \not\varepsilon_2^+ \not k_1 \not k_2]}{\varepsilon_1^+ \cdot k_2\ \varepsilon_2^+ \cdot k_1} \tag{B.14}$$

$$= \frac{\left\{\mathrm{tr}\left[\frac{1}{2}(1-\gamma_5)\gamma_{\mu_1}\gamma_{\mu_2}\not k_1 \not k_2\right] - \mathrm{tr}\left[\frac{1}{2}(1+\gamma_5)\gamma_{\mu_1}\gamma_{\mu_2}\not k_1 \not k_2\right]\right\} \langle \xi^-|\gamma^{\mu_1}|k_1^-\rangle \langle \xi^-|\gamma^{\mu_2}|k_2^-\rangle}{4\ \langle \xi\, 2\rangle\, [2\, 1]\ \langle \xi\, 1\rangle\, [1\, 2]}\,. \tag{B.15}$$

Fierzing the $\langle \xi^-|\gamma^{\mu_i}|k_i^-\rangle$ strings into the trace gives

$$\begin{aligned} R^{++} &= \frac{\langle \xi\, \xi\rangle\, [2\, 1]\, \langle 1\, 2\rangle\, [2\, 1] - [1\, 2]\, \langle \xi\, 1\rangle\, [1\, 2]\, \langle 2\, \xi\rangle}{\langle \xi\, 2\rangle\, [2\, 1]\ \langle \xi\, 1\rangle\, [1\, 2]} \\ &= -1\,. \end{aligned} \tag{B.16}$$

Repeating the analysis for two negative-helicity gluons yields the opposite sign,

$$\begin{aligned} R^{--} &\equiv \frac{i\varepsilon_{\mu_1\mu_2\nu_1\nu_2}\varepsilon^{-,\mu_1}\varepsilon^{-,\mu_2}k_1^{\nu_1}k_2^{\nu_2}}{-\varepsilon_1^-\cdot k_2\ \varepsilon_2^-\cdot k_1} \\ &= +1\,. \end{aligned} \tag{B.17}$$

Thus the second and third terms cancel in the positive-helicity case, so $A_2(\phi, 1^+, 2^+) = 0$; whereas they add in the negative-helicity case, for which one finds eq. (2.9).

To compute $A_n(\phi, 1^\pm, 2^+, 3^+, \ldots, n^+)$ using the Berends-Giele off-shell currents, we merely join each gluon produced by the vertices from $\phi\, \mathrm{tr}\, G^{SD\,\mu\nu} G_{SD}^{\mu\nu}$ to an off-shell current. The two currents we need are [50, 58]

$$J_{1,n}^{+,\mu} \equiv J^\mu(1^+, 2^+, \ldots, n^+) = \frac{\langle \xi^- | \gamma^\mu\ \not{P}_{1,n} | \xi^+ \rangle}{\sqrt{2}\, \langle \xi\, 1\rangle\, \langle 1\, 2\rangle \cdots \langle n-1,\, n\rangle\, \langle n\, \xi\rangle}\,, \tag{B.18}$$

where all reference momenta are taken to be equal to ξ, and

$$J_{1,n}^{-,\mu} \equiv J^\mu(1^-, 2^+, \ldots, n^+) = \frac{\langle 1^- | \gamma^\mu\ \not{P}_{1,n} | 1^+ \rangle}{\sqrt{2}\, \langle 1\, 2\rangle \cdots \langle n\, 1\rangle} \sum_{m=3}^{n} \frac{\langle 1^- | \not{k}_m\ \not{P}_{1,m} | 1^+ \rangle}{P_{1,m-1}^2 P_{1,m}^2}\,, \tag{B.19}$$

where the reference momentum choice is $\xi_1 = k_2$, $\xi_2 = \cdots = \xi_n = k_1$. In these formulae, $\not{P}_{p,q} = k_p + k_{p+1} + \ldots + k_{q-1} + k_q$.

Actually, the current (B.19) is not quite sufficient for the proof in the one-minus case. We really need the current where the negative-helicity gluon appears at an arbitrary position in the chain of positive-helicity gluons (all with the same reference momentum), $J^\mu(2^+, 3^+, \ldots, 1^-, \ldots, n^+)$. This current has been constructed by Mahlon [26]. The expression is rather complicated, so we do not present it here. It is sufficient for our purposes to note that it is also proportional to $\langle 1^- | \gamma^\mu\ \not{P}_{1,n} | 1^+ \rangle$.

For $A_n(\phi, 1^+, 2^+, 3^+, \ldots, n^+)$, we take all reference momenta equal to ξ, a generic vector. For $A_n(\phi, 1^-, 2^+, 3^+, \ldots, n^+)$, we take $\xi_1 = k_2$, $\xi_2 = \cdots = \xi_n = k_1 \equiv \xi$. Then in both cases all the currents attaching to the Higgs vertex are proportional to $\langle \xi^- | \gamma^\mu \ldots$. In terms of spinor notation, all currents are proportional to ξ_α. This property is all we need to demonstrate (again via Fierz identities) that

$$J^+ \cdot J^+ = J^+ \cdot J^- = 0\,, \tag{B.20}$$

$$\varepsilon_{\mu_1\mu_2\mu_3\mu_4} J^{+,\mu_1} J^{+,\mu_2} J^{\pm,\mu_3} = 0\,. \tag{B.21}$$

These relations in turn suffice to show that the Feynman vertices coupling ϕ to 3 or 4 gluons, ϕggg and $\phi gggg$, do not contribute to $A_n(\phi, 1^\pm, 2^+, 3^+, \ldots, n^+)$. Terms in these vertices without a Levi-Civita tensor always attach a Minkowski metric $\eta_{\mu_1\mu_2}$ to two currents; their contribution vanishes according to eq. (B.20). (The same is true of the first term in the ϕgg vertex (B.13).) Terms containing the Levi-Civita tensor $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ attach it directly to at least three currents; their contribution vanishes according to eq. (B.21). This leaves just the contributions of the second and third terms in the ϕgg vertex (B.13). They cancel against each other, just as in the case of $A_2(\phi, 1^+, 2^+)$ above. Suppose that gluons $p+1$ through m (cyclicly) attach to one leg of the ϕgg vertex, and gluons $m+1$ through p

(cyclicly) attach to the other leg. Then the ratio analogous to eq. (B.14) is

$$R^{*,\pm+}_{p,m} \equiv \frac{i\varepsilon_{\mu_1\mu_2\nu_1\nu_2}\langle\xi^-|\gamma^{\mu_1}\,\not{P}_{p+1,m}|\xi^+\rangle\langle\xi^-|\gamma^{\mu_2}\,\not{P}_{m+1,p}|\xi^+\rangle k_1^{\nu_1}k_2^{\nu_2}}{-\langle\xi^-|\,\not{P}_{m+1,p}\,\not{P}_{p+1,m}|\xi^+\rangle\langle\xi^-|\,\not{P}_{p+1,m}\,\not{P}_{m+1,p}|\xi^+\rangle} \tag{B.22}$$

$$= -1\,, \tag{B.23}$$

using the same Fierz identities as before. This completes the recursive proof that

$$A_n(\phi, 1^\pm, 2^+, 3^+, \ldots, n^+) = 0\,. \tag{B.24}$$

B.3 Recursive construction of $A_n(\phi^\dagger, 1^+, 2^+, \ldots, n^+)$

Using the same all-plus current (B.18), but flipping the sign of the Levi-Civita term in the Higgs vertex (B.13), we can give a recursive construction of the non-vanishing all-plus amplitudes $A_n(\phi^\dagger, 1^+, 2^+, \ldots, n^+)$. Note that from the point of view of MHV rules, these amplitudes are 'maximally googly'.

The only difference from the above analysis of $A_n(\phi, 1^+, 2^+, 3^+, \ldots, n^+)$ is that now the ratio $R^{*,++}_{p,m}$ in eq. (B.23) becomes equal to $+1$, so the terms coming from the second and third terms in the vertex (B.13) add instead of cancelling. Restoring the overall factors,

$$A_n^{\dagger+\cdots+} \equiv A_n(\phi^\dagger, 1^+, 2^+, \ldots, n^+) = -\frac{1}{\rho}\sum_{1\le m<p\le n}\frac{\langle m,\, m+1\rangle}{\langle m\,\xi\rangle\,\langle \xi,\, m+1\rangle}\frac{\langle p,\, p+1\rangle}{\langle p\,\xi\rangle\,\langle \xi,\, p+1\rangle}\times$$
$$\times\langle\xi^-|\,\not{P}_{p+1,m}\,\not{P}_{m+1,p}|\xi^+\rangle^2\,, \tag{B.25}$$

where $\rho = \langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle$. We can replace $\not{P}_{p+1,m}$ by $\not{P}_{1,n}$, and then expand out $\not{P}_{m+1,p}$ to get,

$$A_n^{\dagger+\cdots+} = -\frac{1}{\rho}\sum_{1\le m<(k,l)\le p\le n}\frac{\langle m,\, m+1\rangle}{\langle m\,\xi\rangle\,\langle \xi,\, m+1\rangle}\frac{\langle p,\, p+1\rangle}{\langle p\,\xi\rangle\,\langle \xi,\, p+1\rangle}\langle\xi^-|\,\not{P}_{1,n}\not{k}|\xi^+\rangle\langle\xi^-|\,\not{P}_{1,n}\not{l}|\xi^+\rangle\,. \tag{B.26}$$

We use the $k \leftrightarrow l$ symmetry to write the sum over k and l as twice the sum over $m < k < l \le p$, plus the (diagonal) sum over $m < k = l \le p$. We then apply the eikonal identity,

$$\sum_{m=j}^{k-1}\frac{\langle m,\, m+1\rangle}{\langle m\,\xi\rangle\,\langle \xi,\, m+1\rangle} = \frac{\langle j\,k\rangle}{\langle j\,\xi\rangle\,\langle \xi\,k\rangle}\,, \tag{B.27}$$

in order to carry out the sums over m and p. After rearranging some of the spinor products, we have

$$\begin{aligned}
A_n^{\dagger+\cdots+} &= \frac{1}{\rho\langle\xi\,1\rangle^2}\left\{2\sum_{1\le k<l\le n}\langle\xi^-|\,\not{P}_{1,n}\not{k}|1^+\rangle\langle\xi^-|\,\not{P}_{1,n}\not{l}|1^+\rangle + \sum_{1\le k\le n}\langle\xi^-|\,\not{P}_{1,n}\not{k}|1^+\rangle^2\right\}\\
&= \frac{1}{\rho\langle\xi\,1\rangle^2}\langle\xi^-|\,\not{P}_{1,n}\,\not{P}_{1,n}|1^+\rangle^2\\
&= \frac{P_{1,n}^2}{\rho}\\
&= \frac{m_H^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,,
\end{aligned} \tag{B.28}$$

as desired. Since the corresponding ϕ amplitude vanishes, eq. (B.28) is also the result for the all-plus Higgs amplitude, eq. (1.2).

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JHEP12(2004)015

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